ta07-01, ha07-01

One ticket to a show costs \$20 at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 15 **(E)** 20

2007 AMC 10 A, Problem #1—
2007 AMC 12 A, Problem #1—
"Find the total cost for Susan and Pam"

Solution

Answer (C): Susan pays $(4)(0.75)(20) = 60$ dollars. Pam pays $(5)(0.70)(20) = 70$ dollars, so she pays $70 - 60 = 10$ more dollars than Susan.

Difficulty: Easy
NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

tb07-01, hb07-01

Isabella's house has 3 bedrooms. Each bedroom is 12 feet long, 10 feet wide, and 8 feet high. Isabella must paint the walls of all the bedrooms. Doorways and windows, which will not be painted, occupy 60 square feet in each bedroom. How many square feet of walls must be painted?

(A) 678 **(B)** 768 **(C)** 786 **(D)** 867 **(E)** 876

2007 AMC 10 B, Problem #1—
2007 AMC 12 B, Problem #1—
"Paint only the walls, that means no ceilings and floors."

Solution

Answer (E): The perimeter of each bedroom is $2(12+10) = 44$ feet, so the surface to be painted in each bedroom has an area of $44 \cdot 8 - 60 = 292$ square feet. Since there are 3 bedrooms, Isabella must paint $3 \cdot 292 = 876$ square feet.

Difficulty: Medium-easy
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Cubes

tb07-02

Define the operation $\star$ by $a \star b = (a+b)b$. What is $(3 \star 5) - (5 \star 3)$?

(A) -16 **(B)** -8 **(C)** 0 **(D)** 8 **(E)** 16

2007 AMC 10 B, Problem #2— "Plug in 3, 5 for a, b."

Solution

Answer (E): Since $3 \star 5 = (3+5)5 = 8 \cdot 5 = 40$ and $5 \star 3 = (5+3)3 = 8 \cdot 3 = 24$, we have

$$3 \star 5 - 5 \star 3 = 40 - 24 = 16.$$

Difficulty: Medium-easy
NCTM Standard: Number and Operations Standard: Understand meanings of operations and how they relate to one another.
Mathworld.com Classification: Algebra > Algebraic Operations > General Algebraic Operations > Binary Operation

ta07-03, ha07-02

An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

(A) 0.5 **(B)** 1 **(C)** 1.5 **(D)** 2 **(E)** 2.5

2007 AMC 10 A, Problem #3—
2007 AMC 12 A, Problem #2—
"The additional volume of water occupied in the aquarium is equal to the volume of the brick."

Solution

Answer (D): The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have

$$8000 = 4000h \quad \text{and} \quad h = 2 \text{ centimeters}$$

Difficulty: Medium
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Solid Geometry > Volume

ha07-04

Kate rode her bicycle for 30 minutes at a speed of 16 mph, then walked for 90 minutes at a speed of 4 mph. What was her overall average speed in miles per hour?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 14

2007 AMC 12 A, Problem #4— "Find the total distance Kate travels in $30 + 90 = 120$ minutes $= 2$ hours "

Solution

Answer (A): Kate rode for 30 minutes = 1/2 hour at 16 mph, so she rode 8 miles. She walked for 90 minutes = 3/2 hours at 4 mph, so she walked 6 miles. Therefore she covered a total of 14 miles in 2 hours, so her average speed was 7 mph.

Difficulty: Medium-easy
NCTM Standard: Problem Solving Standard: apply and adapt a variety of appropriate strategies to solve problems.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Proportional

hb07-04

At Frank's Fruit Market, 3 bananas cost as much as 2 apples, and 6 apples cost as much as 4 oranges. How many oranges cost as much as 18 bananas?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 18

2007 AMC 12 B, Problem #4— "18 bananas cost as much as 12 apples."

Solution

Answer (B): Because 3 bananas cost as much as 2 apples, 18 bananas cost as much as 12 apples. Because 6 apples cost as much as 4 oranges, 12 apples cost as much as 8 oranges. Therefore 18 bananas cost as much as 8 oranges.

Difficulty: Easy
NCTM Standard: Number and Operations Standard: Understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Proportional

ta07-04, ha07-03

The larger of two consecutive odd integers is three times the smaller. What is their sum?

(A) 4 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 20

2007 AMC 10 A, Problem #4—
2007 AMC 12 A, Problem #3—
"Set up an equation to represent the relation of the two integers."

Solution

Answer (A): Let the smaller of the integers be x. Then the larger is $x+2$. So $x+2=3x$, from which $x=1$. Thus the two integers are 1 and 3, and their sum is 4.

Difficulty: Easy
NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Number Theory > Integers > Odd number

ta07-05

A school store sells 7 pencils and 8 notebooks for \$4.15. It also sells 5 pencils and 3 notebooks for \$1.77. How much do 16 pencils and 10 notebooks cost?

(A) \$4.76 **(B)** \$5.84 **(C)** \$6.00 **(D)** \$6.16 **(E)** \$6.32

2007 AMC 10 A, Problem #5— "Set up two equations to solve for the cost of a pencil and a notebook."

Solution

Answer (B): Let p be the cost in cents of a pencil and n be the cost in cents of a notebook. Then

$$7p + 8n = 415 \quad \text{and} \quad 5p + 3n = 177.$$

The solution of this pair of equations is $p = 9$ and $n = 44$. So the cost of 16 pencils and 10 notebooks is $16(9) + 10(44) = 584$ cents, or \$5.84.

Difficulty: Medium-easy
NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Algebra > Algebraic Equations > Linear Equation

hb07-06

Triangle ABC has side lengths $AB = 5$, $BC = 6$, and $AC = 7$. Two bugs start simultaneously from A and crawl along the sides of the triangle in opposite directions at the same speed. They meet at point D. What is BD?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2007 AMC 12 B, Problem #6— "Find the perimeter of the triangle."

Solution

Answer (D): The perimeter of the triangle is $5 + 6 + 7 = 18$, so the distance that each bug crawls is 9. Therefore $AB + BD = 9$, and $BD = 4$.

Difficulty: Medium-easy

NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Calculus and Analysis > Differential Geometry > Differential Geometry of Curves > Perimeter

ta07-06

At Euclid High School, the number of students taking the AMC10 was 60 in 2002, 66 in 2003, 70 in 2004, 76 in 2005, and 78 in 2006, and is 85 in 2007. Between what two consecutive years was there the largest percentage increase?

(A) 2002 and 2003 **(B)** 2003 and 2004 **(C)** 2004 and 2005

(D) 2005 and 2006 **(E)** 2006 and 2007

2007 AMC 10 A, Problem #6—
"Find out the percentage increase for each year."

Solution

Answer (A): Between 2002 and 2003, the increase was

$$\frac{6}{60} = \frac{1}{10} = 10\%.$$

Between the other four pairs of consecutive years, the increases were

$$\frac{4}{66} < \frac{4}{40} = \frac{1}{10}, \quad \frac{6}{70} < \frac{6}{60} = \frac{1}{10}, \quad \frac{2}{76} < \frac{2}{20} = \frac{1}{10}, \quad \text{and} \quad \frac{7}{78} < \frac{7}{70} = \frac{1}{10}.$$

Therefore the largest percentage increase occurred between 2002 and 2003.

Difficulty: Medium

NCTM Standard: Algebra Standard: analyze change in various contexts.

Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

ta07-07, ha07-05

Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of \$10,500 for both taxes. How many dollars was the inheritance?

(A) 30,000 **(B)** 32,500 **(C)** 35,000 **(D)** 37,500 **(E)** 40,000

2007 AMC 10 A, Problem #7—
2007 AMC 12 A, Problem #5—
"Find out the percentage of the inheritance paid in taxes."

Solution

Answer (D): After paying the federal taxes, Mr. Public had 80% of his inheritance money left. He paid 10% of that, or 8% of his inheritance, in state taxes. Hence his total tax bill was 28% of his inheritance, and his inheritance was $\$10{,}500/0.28 = \$37{,}500$.

Difficulty: Medium
NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Percent

tb07-07, hb07-07

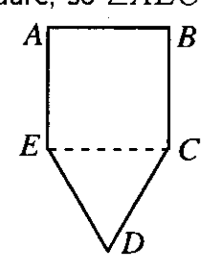
All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90°$. What is the degree measure of $\angle E$?

(A) 90 **(B)** 108 **(C)** 120 **(D)** 144 **(E)** 150

2007 AMC 10 B, Problem #7—
2007 AMC 12 B, Problem #7—
"Quadrilateral $ABCE$ is a square, and $\triangle CDE$ is equilateral."

Solution

Answer (E): Because $AB = BC = EA$ and $\angle A = \angle B = 90°$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90°$.



Also $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60°$. Therefore

$$\angle E = \angle AEC + \angle CED = 90° + 60° = 150°.$$

Difficulty: Medium-hard

NCTM Standard: Geometry Standard for Grades 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Polygons > Convex Polygon
Geometry > Plane Geometry > Polygons > Pentagon

ta07-08, ha07-06

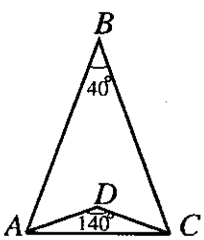
Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40°$, and $\angle ADC = 140°$. What is the degree measure of $\angle BAD$?

(A) 20 **(B)** 30 **(C)** 40 **(D)** 50 **(E)** 60

2007 AMC 10 A, Problem #8— 2007 AMC 12 A, Problem #6— "$\angle BAD = \angle BAC - \angle DAC$."

Solution

Answer (D): Because $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180° - \angle ABC) = 70°$.



Similarly,

$$\angle DAC = \frac{1}{2}(180° - \angle ADC) = 20°.$$

Thus $\angle BAD = \angle BAC - \angle DAC = 50°$.

OR

Because $\triangle ABC$ and $\triangle ADC$ are isosceles triangles and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$, applying the Exterior Angle Theorem to $\triangle ABD$ gives $\angle BAD = 70° - 20° = 50°$.

Difficulty: Medium
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Isosceles Triangles

tb07-08

On the trip home from the meeting where this AMC10 was constructed, the Contest Chair noted that his airport parking receipt had digits of the form $bbcac$, where $0 \leq a < b < c \leq 9$, and b was the average of a and c. How many different five-digit numbers satisfy all these properties?

(A) 12 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

2007 AMC 10 B, Problem #8—

"Once a and c are chosen, the integer b is determined. $a + c$ is even."

Solution

Answer (D): Once a and c are chosen, the integer b is determined. For $a = 0$, we could have $c = 2$, 4, 6, or 8. For $a = 2$, we could have $c = 4$, 6, or 8. For $a = 4$, we could have $c = 6$ or 8, and for $a = 6$ the only possibility is $c = 8$. Thus there are $1 + 2 + 3 + 4 = 10$ possibilities when a is even. Similarly, there are 10 possibilities when a is odd, so the number of possibilities is 20.

Difficulty: Medium-hard

NCTM Standard: Number and Operations Standard: Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

Mathworld.com Classification: Number Theory > Arithmetic > Number Bases > Digit

ta07-09

Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

(A) -60 **(B)** -17 **(C)** 9 **(D)** 12 **(E)** 60

2007 AMC 10 A, Problem #9— "$a = 4(b+2)$ and $3b = a - 3$."

Solution

Answer (E): The given equations are equivalent, respectively, to

$$3^a = 3^{4(b+2)} \quad \text{and} \quad 5^{3b} = 5^{a-3}.$$

Therefore $a = 4(b+2)$ and $3b = a - 3$. The solution of this system is $a = -12$ and $b = -5$, so $ab = 60$.

Difficulty: Medium-hard

NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.

Mathworld.com Classification: Calculus and Analysis > Special Functions > Powers > Exponent

tb07-09

A cryptographic code is designed as follows. The first time a letter appears in a given message it is replaced by the letter that is 1 place to its right in the alphabet (assuming that the letter A is one place to the right of the letter Z). The second time this same letter appears in the given message, it is replaced by the letter that is $1+2$ places to the right, the third time it is replaced by the letter that is $1+2+3$ places to the right, and so on. For example, with this code the word "banana" becomes "cbodqg". What letter will replace the last letter s in the message

"Lee's sis is a Mississippi miss, Chriss!"?

(A) g **(B)** h **(C)** o **(D)** s **(E)** t

2007 AMC 10 B, Problem #9—

"The last s is the 12th appearance of this letter in the message."

Solution

Answer (D): The last s is the 12th appearance of this letter in the message, so it will be replaced by the letter that is

$$1+2+3+\cdots+12=\frac{1}{2}(12\cdot 13)=3\cdot 26$$

letters to the right of s. Since the alphabet has 26 letters, this letter s is coded as s.

Difficulty: Medium-easy
NCTM Standard: Algebra Standard: Represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Recreational Mathematics > Cryptograms

tb07-10

Two points B and C are in a plane. Let S be the set of all points A in the plane for which $\triangle ABC$ has area 1. Which of the following describes S?

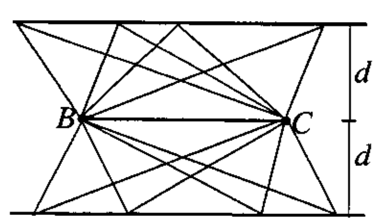
(A) two parallel lines **(B)** a parabola **(C)** a circle **(D)** a line segment **(E)** two points

2007 AMC 10 B, Problem #10—

"Since points B and C are fixed, the altitude from A must be constant if $\triangle ABC$ has area 1."

Solution

Answer (A): If the altitude from A has length d, then $\triangle ABC$ has area $(1/2)(BC)d$. The area is 1 if and only if $d = 2/(BC)$. Thus S consists of the two lines that are parallel to line BC and are $2/(BC)$ units from it, as shown.



Difficulty: Hard

NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Miscellaneous Plane Geometry > Area

ta07-11

The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

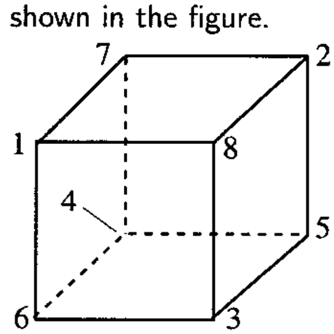
2007 AMC 10 A, Problem #11— "Each vertex appears on exactly three faces."

Solution

Answer (C): Each vertex appears on exactly three faces, so the sum of the numbers on all the faces is

$$3(1+2+\cdots+8) = 3 \cdot \frac{8 \cdot 9}{2} = 108.$$

There are six faces for the cube, so the common sum must be $108/6 = 18$. A possible numbering is shown in the figure.



Difficulty: Medium
NCTM Standard: Geometry Standard: apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Cubes

tb07-12, hb07-08

Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

2007 AMC 10 B, Problem #12—
2007 AMC 12 B, Problem #8—
"The sum of his three children's ages N years ago was $T - 3N$."

Solution

Answer (D): Tom's age N years ago was $T - N$. The sum of his three children's ages at that time was $T - 3N$. Therefore $T - N = 2(T - 3N)$, so $5N = T$ and $T/N = 5$. The conditions of the problem can be met, for example, if Tom's age is 30 and the ages of his children are 9, 10, and 11. In that case $T = 30$ and $N = 6$.

Difficulty: Hard
NCTM Standard: Number and Operations Standard for Grade 9-12: Understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Number Theory > Arithmetic > Addition and Subtraction
Number Theory > Arithmetic > Fractions > Proportional

ta07-12

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

2007 AMC 10 A, Problem #12—

"The first guide can take any combination of tourists except all the tourists or none of the tourists. "

Solution

Answer (D): The first guide can take any combination of tourists except all the tourists or none of the tourists. Therefore the number of possibilities is

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62.$$

OR

If each guide did not need to take at least one tourist, then each tourist could choose one of the two guides independently. In this case there would be $2^6 = 64$ possible arrangements. The two arrangements for which all tourists choose the same guide must be excluded, leaving a total of $64 - 2 = 62$ possible arrangements.

Difficulty: Hard
NCTM Standard: Problem Solving Standard: apply and adapt a variety of appropriate strategies to solve problems.
Mathworld.com Classification: Discrete Mathematics > Combinatorics > Binomial Coefficients

ta07-13, ha07-09

Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{6}{7}$

2007 AMC 10 A, Problem #13—
2007 AMC 12 A, Problem #9—
"Set up an equation on time, and find the ratio from the equation."

Solution

Answer (B): Let w be Yan's walking speed, and let x and y be the distances from Yan to his home and to the stadium, respectively. The time required for Yan to walk to the stadium is y/w, and the time required for him to walk home is x/w. Because he rides his bicycle at a speed of $7w$, the time required for him to ride his bicycle from his home to the stadium is $(x+y)/(7w)$. Thus

$$\frac{y}{w} = \frac{x}{w} + \frac{x+y}{7w} = \frac{8x+y}{7w}.$$

As a consequence, $7y = 8x + y$, so $8x = 6y$. The required ratio is $x/y = 6/8 = 3/4$.

OR

Because we are interested only in the ratio of the distances, we may assume that the distance from Yan's home to the stadium is 1 mile. Let x be his present distance from his home. Imagine that Yan has a twin, Nay. While Yan walks to the stadium, Nay walks to their home and continues $1/7$ of a mile past their home. Because walking $1/7$ of a mile requires the same amount of time as riding 1 mile, Yan and Nay will complete their trips at the same time. Yan has walked $1 - x$ miles while Nay has walked $x + \frac{1}{7}$ miles, so $1 - x = x + \frac{1}{7}$. Thus $x = 3/7$, $1 - x = 4/7$, and the required ratio is $x/(1-x) = 3/4$.

Difficulty: Medium-hard
NCTM Standard: Problem Solving Standard: apply and adapt a variety of appropriate strategies to solve problems.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Ratio

tb07-13

Two circles of radius 2 are centered at $(2,0)$ and at $(0,2)$. What is the area of the intersection of the interiors of the two circles?

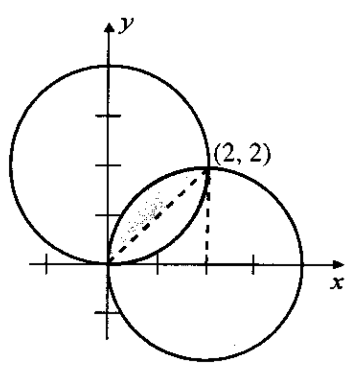
(A) $\pi - 2$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi\sqrt{3}}{3}$ **(D)** $2(\pi - 2)$ **(E)** π

2007 AMC 10 B, Problem #13—

"Half of the region is formed by removing an isosceles right triangle of leg length 2 from a quarter of one of the circles."

Solution

Answer (D): The two circles intersect at $(0,0)$ and $(2,2)$, as shown.



Half of the region described is formed by removing an isosceles right triangle of leg length 2 from a quarter of one of the circles. Because the quarter-circle has area $(1/4)\pi(2)^2 = \pi$ and the triangle has area $(1/2)(2)^2 = 2$, the area of the region is $2(\pi - 2)$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.
Mathworld.com Classification: Geometry > Plane Geometry > Circles

ta07-14, ha07-10

A triangle with side lengths in the ratio 3:4:5 is inscribed in a circle of radius 3. What is the area of the triangle?

(A) 8.64 **(B)** 12 **(C)** 5π **(D)** 17.28 **(E)** 18

2007 AMC 10 A, Problem #14—
2007 AMC 12 A, Problem #10—
"The hypotenuse of the triangle is the diameter of the circle."

Solution

Answer (A): Let the sides of the triangle have lengths $3x$, $4x$, and $5x$. The triangle is a right triangle, so its hypotenuse is a diameter of the circle. Thus $5x = 2 \cdot 3 = 6$, so $x = 6/5$. The area of the triangle is

$$\frac{1}{2} \cdot 3x \cdot 4x = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

OR

A right triangle with side lengths 3, 4, and 5 has area $(1/2)(3)(4) = 6$. Because the given right triangle is inscribed in a circle with diameter 6, the hypotenuse of this triangle has length 6. Thus the sides of the given triangle are $6/5$ as long as those of a $3-4-5$ triangle, and its area is $(6/5)^2$ times that of a $3-4-5$ triangle. The area of the given triangle is

$$\left(\frac{6}{5}\right)^2 (6) = \frac{216}{25} = 8.64.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Right Triangle

tb07-14, hb07-10

Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

2007 AMC 10 B, Problem #14—
2007 AMC 12 B, Problem #10—
"Set up two equations to represent above two conditions."

Solution

Answer (C): Let g be the number of girls and b the number of boys initially in the group. Then $g = 0.4(g + b)$. After two girls leave and two boys arrive, the size of the entire group is unchanged, so $g-2 = 0.3(g+b)$. The solution of the system of equations

$$g = 0.4(g + b) \quad \text{and} \quad g - 2 = 0.3(g + b)$$

is $g = 8$ and $b = 12$, so there were initially 8 girls.

OR

After two girls leave and two boys arrive, the size of the group is unchanged. So the two girls who left represent $40\% - 30\% = 10\%$ of the group. Thus the size of the group is 20, and the original number of girls was 40% of 20, or 8.

Difficulty: Medium-easy
NCTM Standard: Data Analysis and Probability Standard for Grade 9-12: Develop and evaluate inferences and predictions that are based on data.
Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Proportional

hb07-15

The geometric series $a + ar + ar^2 + \cdots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$?

(A) $\frac{4}{3}$ **(B)** $\frac{12}{7}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{7}{3}$ **(E)** $\frac{5}{2}$

2007 AMC 12 B, Problem #15—
"$a + ar + ar^2 + \cdots = \frac{a}{1-r}$."

Solution

Answer (E): The terms involving odd powers of r form the geometric series $ar + ar^3 + ar^5 + \cdots$. Thus

$$7 = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

and

$$3 = ar + ar^3 + ar^5 + \cdots = \frac{ar}{1-r^2} = \frac{a}{1-r} \cdot \frac{r}{1+r} = \frac{7r}{1+r}.$$

Therefore $r = 3/4$. It follows that $a/(1/4) = 7$, so $a = 7/4$ and

$$a + r = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}.$$

OR

The sum of the terms involving even powers of r is $7 - 3 = 4$. Therefore

$$3 = ar + ar^3 + ar^5 + \cdots = r(a + ar^2 + ar^4 + \cdots) = 4r,$$

so $r = 3/4$. As in the first solution, $a = 7/4$ and $a + r = 5/2$.

Difficulty: Hard
NCTM Standard: Algebra Standard for Grade 9-12: Understand patterns, relations, and functions.
Mathworld.com Classification: Calculus and Analysis > Series > General Series > Geometric Series

tb07-15, hb07-11

The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number?

(A) 125 **(B)** 144 **(C)** 153 **(D)** 173 **(E)** 180

2007 AMC 10 B, Problem #15—
2007 AMC 12 B, Problem #11—
"The degree measures all four angles of quadrilateral $ABCD$ have a sum of 360."

Solution

Answer (D): Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

Difficulty: Medium-hard
NCTM Standard: Geometry Standard for Grade 9-12: analyze properties and determine attributes of two- and three-dimensional objects.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals

ha07-16

How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?

(A) 96 **(B)** 104 **(C)** 112 **(D)** 120 **(E)** 256

2007 AMC 12 A, Problem #16—

"The set of the three digits of such a number can be arranged to form an increasing arithmetic sequence."

Solution

Answer (C): The set of the three digits of such a number can be arranged to form an increasing arithmetic sequence. There are 8 possible sequences with a common difference of 1, since the first term can be any of the digits 0 through 7. There are 6 possible sequences with a common difference of 2, 4 with a common difference of 3, and 2 with a common difference of 4. Hence there are 20 possible arithmetic sequences. Each of the 4 sets that contain 0 can be arranged to form $2 \cdot 2! = 4$ different numbers, and the 16 sets that do not contain 0 can be arranged to form $3! = 6$ different numbers. Thus there are a total of $4 \cdot 4 + 16 \cdot 6 = 112$ numbers with the required properties.

Difficulty: Hard
NCTM Standard: Number and Operations Standard: understand numbers, ways of representing numbers, relationships among numbers, and number systems.
Mathworld.com Classification: Calculus and Analysis > Series > General Series > Arithmetic Series

hb07-16

Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?

(A) 15 **(B)** 18 **(C)** 27 **(D)** 54 **(E)** 81

2007 AMC 12 B, Problem #16—

"There is only one coloring for each ordered triple, $(red, white, blue)$."

Solution

Answer (A): Let r, w, and b be the number of red, white, and blue faces, respectively. Then (r, w, b) is one of 15 possible ordered triples, namely one of the three permutations of $(4,0,0)$, $(2,2,0)$, or $(2,1,1)$, or one of the six permutations of $(3,1,0)$. The number of distinguishable colorings for each of these ordered triples is the same as for any of its permutations. If $(r, w, b) = (4,0,0)$, then exactly one coloring is possible. If $(r, w, b) = (3,1,0)$, the tetrahedron can be placed with the white face down. If $(r, w, b) = (2,2,0)$, the tetrahedron can be placed with one white face down and the other facing forward. If $(r, w, b) = (2,1,1)$, the tetrahedron can be placed with the white face down and the blue face forward. Therefore there is only one coloring for each ordered triple, and the total number of distinguishable colorings is 15.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: analyze properties and determine attributes of two- and three-dimensional objects.
Mathworld.com Classification: Discrete Mathematics > Graph Theory > Labeled Graphs > Coloring
Geometry > Transformations > Rotations

tb07-16, hb07-12

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?

(A) 85 **(B)** 88 **(C)** 93 **(D)** 94 **(E)** 98

2007 AMC 10 B, Problem #16—
2007 AMC 12 B, Problem #12—
"Set up a equation for total points the class scored."

Solution

Answer (C): Let N be the number of students in the class. Then there are $0.1N$ juniors and $0.9N$ seniors. Let s be the score of each junior. The scores totaled $84N = 83(0.9N) + s(0.1N)$, so

$$s = \frac{84N - 83(0.9N)}{0.1N} = 93.$$

Note: In this problem, we could assume that the class has one junior and nine seniors. Then

$$9 \cdot 83 + s = 10 \cdot 84 = 9 \cdot 84 + 84 \quad \text{and} \quad s = 9(84 - 83) + 84 = 93.$$

Difficulty: Medium

NCTM Standard: Data Analysis and Probability Standard for Grades 9-12: Develop and evaluate inferences and predictions that are based on data.

Mathworld.com Classification: Number Theory > Arithmetic > Fractions > Proportional

ha07-17

Suppose that $\sin a+\sin b = \sqrt{5/3}$ and $\cos a+\cos b = 1$. What is $\cos(a-b)$?

(A) $\sqrt{\frac{5}{3}}-1$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** 1

2007 AMC 12 A, Problem #17—

"$\cos(a-b) = \sin a \sin b + \cos a \cos b$."

Solution

Answer (B): Square both sides of both given equations to obtain

$\sin^2 a+2\sin a \sin b+\sin^2 b = 5/3$ and $\cos^2 a+2\cos a\cos b+\cos^2 b = 1.$

Then add corresponding sides of the resulting equations to obtain

$$(\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) + 2(\sin a \sin b + \cos a \cos b) = \frac{8}{3}.$$

Because $\sin^2 a + \cos^2 a = \sin^2 b + \cos^2 b = 1$, it follows that

$$\cos(a-b) = \sin a \sin b + \cos a \cos b = \frac{1}{3}.$$

One ordered pair (a, b) that satisfies the given condition is approximately $(0.296, 1.527)$.

Difficulty: Hard
NCTM Standard: Algebra Standard for Grade 9-12: write equivalent forms of equations, inequalities, and systems of equations and solve them with fluencymentally or with paper and pencil in simple cases and using technology in all cases.
Mathworld.com Classification: Geometry > Trigonometry > General Trigonometry

ta07-17

Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

(A) 15 **(B)** 30 **(C)** 50 **(D)** 60 **(E)** 5700

2007 AMC 10 A, Problem #17—

"An integer is a cube if and only if, in the prime factorization of the number, each prime factor occurs a multiple of three times."

Solution

Answer (D): An integer is a cube if and only if, in the prime factorization of the number, each prime factor occurs a multiple of three times. Because $n^3 = 75m = 3 \cdot 5^2 \cdot m$, the minimum value for m is $3^2 \cdot 5 = 45$. In that case $n = 15$, and $m + n = 60$.

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Calculus and Analysis > Special Functions > Powers > Cubed

ha07-18

The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

(A) 0 **(B)** 1 **(C)** 4 **(D)** 9 **(E)** 16

2007 AMC 12 A, Problem #18—
"Because $f(x)$ has real coefficients and $2i$ and $2+i$ are zeros, so are their conjugates $-2i$ and $2-i$."

Solution

Answer (D): Because $f(x)$ has real coefficients and $2i$ and $2+i$ are zeros, so are their conjugates $-2i$ and $2-i$. Therefore

$$\begin{aligned} f(x) = (x+2i)(x-2i)(x-(2+i))(x-(2-i)) &= (x^2+4)(x^2-4x+5) \\ &= x^4 - 4x^3 + 9x^2 - 16x + 20. \end{aligned}$$

Hence $a+b+c+d = -4+9-16+20 = 9$.

OR

As in the first solution,

$$f(x) = (x+2i)(x-2i)(x-(2+i))(x-(2-i)),$$

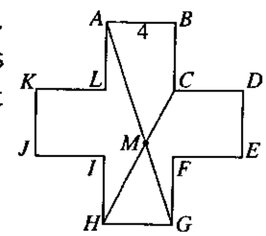
so

$$a+b+c+d = f(1)-1 = (1+2i)(1-2i)(-1-i)(-1+i)-1 = (1+4)(1+1)-1 = 9.$$

Difficulty: Hard
NCTM Standard: Algebra Standard for Grade 9-12: understand relations and functions and select, convert flexibly among, and use various representations for them.
Mathworld.com Classification: Algebra > Polynomials

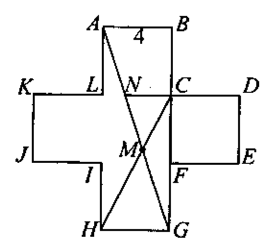
ta07-18

Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



(A) $\frac{44}{3}$ **(B)** 16 **(C)** $\frac{88}{5}$ **(D)** 20 **(E)** $\frac{62}{3}$

2007 AMC 10 A, Problem #18—
"Extend $\overline{CD}$ past C to meet $\overline{AG}$ at N."



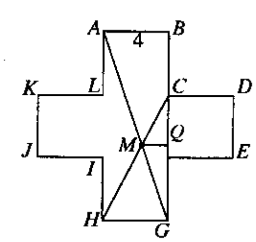
Solution

Answer (C): Extend $\overline{CD}$ past C to meet $\overline{AG}$ at N.
Since $\triangle ABG$ is similar to $\triangle NCG$, $NC = AB \cdot \frac{CG}{BG} = 4 \cdot \frac{8}{12} = \frac{8}{3}$.
This implies that trapezoid $ABCN$ has area $\frac{1}{2} \cdot \left(\frac{8}{3} + 4\right) \cdot 4 = \frac{40}{3}$.
Let v denote the length of the perpendicular from M to $\overline{NC}$. Since $\triangle CMN$ is similar to $\triangle HMG$, and $\frac{GH}{NC} = \frac{4}{8/3} = \frac{3}{2}$,
the length of the perpendicular from M to $\overline{HG}$ is $\frac{3}{2}v$. Because $v + \frac{3}{2}v = 8$, we have $v = \frac{16}{5}$.
Hence the area of $\triangle CMN$ is $\frac{1}{2} \cdot \frac{8}{3} \cdot \frac{16}{5} = \frac{64}{15}$.
So

$$\text{Area}(ABCM) = \text{Area}(ABCN) + \text{Area}(\triangle CMN) = \frac{40}{3} + \frac{64}{15} = \frac{88}{5}.$$

OR

Let Q be the foot of the perpendicular from M to $\overline{BG}$.
Since $\triangle MQG$ is similar to $\triangle ABG$, we have $\frac{MQ}{QG} = \frac{AB}{BG} = \frac{4}{12} = \frac{1}{3}$.
Also, $\triangle MCQ$ is similar to $\triangle HCG$, so $\frac{MQ}{CQ} = \frac{HG}{CG} = \frac{4}{8} = \frac{1}{2}$.
Thus $QG = 3MQ = 3\left(\frac{1}{2}CQ\right) = \frac{3}{2}(8 - QG)$,
which implies that $QG = \frac{24}{5}$ and $MQ = \frac{1}{3}QG = \frac{8}{5}$.
Hence

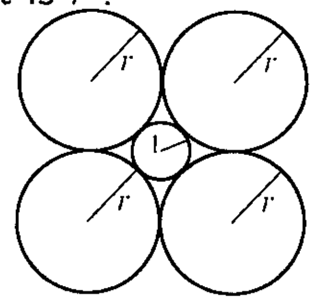


$$\text{Area}(ABCM) = \text{Area}(\triangle ABG) - \text{Area}(\triangle CMG) = \frac{1}{2} \cdot 4 \cdot 12 - \frac{1}{2} \cdot 8 \cdot \frac{8}{5} = \frac{88}{5}.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals

tb07-18

A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r?



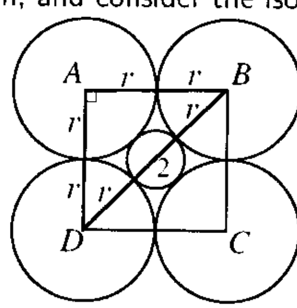
(A) $\sqrt{2}$ **(B)** $1+\sqrt{2}$ **(C)** $\sqrt{6}$ **(D)** 3 **(E)** $2+\sqrt{2}$

2007 AMC 10 B, Problem #18—

"Construct the square $ABCD$ by connecting the centers of the large circles."

Solution

Answer (B): Construct the square $ABCD$ by connecting the centers of the large circles, as shown, and consider the isosceles right $\triangle BAD$.



Since $AB = AD = 2r$ and $BD = 2 + 2r$, we have $2(2r)^2 = (2+2r)^2$. So

$$1 + 2r + r^2 = 2r^2, \quad \text{and} \quad r^2 - 2r - 1 = 0.$$

Applying the quadratic formula gives $r = 1 + \sqrt{2}$.

Difficulty: Hard

NCTM Standard: Geometry Standard for Grades 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Algebra > Algebraic Equations > Quadratic Formula
Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Isosceles Right Triangle

hb07-19

Rhombus $ABCD$, with side length 6, is rolled to form a cylinder of volume 6 by taping $\overline{AB}$ to $\overline{DC}$. What is $\sin(\angle ABC)$?

(A) $\frac{\pi}{9}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\sqrt{3}}{2}$

2007 AMC 12 B, Problem #19—
"The base of the cylinder is a circle with circumference 6, and the height of the cylinder is the altitude of the rhombus."

Solution

Answer (A): Let $\theta = \angle ABC$. The base of the cylinder is a circle with circumference 6, so the radius of the base is $6/(2\pi) = 3/\pi$. The height of the cylinder is the altitude of the rhombus, which is $6\sin\theta$. Thus the volume of the cylinder is

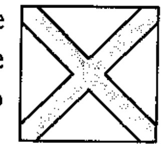
$$6 = \pi\left(\frac{3}{\pi}\right)^2 (6\sin\theta) = \frac{54}{\pi}\sin\theta,$$

so $\sin\theta = \pi/9$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals > Rhombus
Geometry > Solid Geometry > Cylinders

ta07-19

A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?



(A) $2\sqrt{2}+1$ **(B)** $3\sqrt{2}$ **(C)** $2\sqrt{2}+2$ **(D)** $3\sqrt{2}+1$ **(E)** $3\sqrt{2}+2$

2007 AMC 10 A, Problem #19—

"The leg length plus the brush width is equal to half the diagonal of the square."

Solution

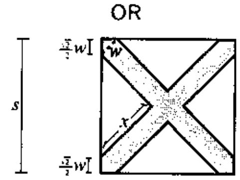
Answer (C): Let s be the side length of the square, let w be the width of the brush, and let x be the leg length of one of the congruent unpainted isosceles right triangles. Since the unpainted area is half the area of the square, the area of each unpainted triangle is 1/8 of the area of the square. So

$$\frac{1}{2}x^2 = \frac{1}{8}s^2 \quad \text{and} \quad x = \frac{1}{2}s.$$

The leg length x plus the brush width w is equal to half the diagonal of the square, so $x+w = (\sqrt{2}/2)s$. Thus

$$w = \frac{\sqrt{2}}{2}s - \frac{1}{2}s \quad \text{and} \quad \frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

OR



The painted stripes have isosceles right triangles with hypotenuse w at each vertex of the square, and the legs of these triangles have length $(\sqrt{2}/2)w$. Since the total area of the four congruent unpainted triangles is half the area of the original square, we have

$$s - \sqrt{2}w = \frac{s}{\sqrt{2}}, \quad \text{so} \quad \sqrt{2}s - 2w = s.$$

and

$$\frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

Difficulty: Hard

NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.

Mathworld.com Classification: Geometry > Plane Geometry > Squares

ha07-20

Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

(A) $\frac{5\sqrt{2}-7}{3}$ (B) $\frac{10-7\sqrt{2}}{3}$ (C) $\frac{3-2\sqrt{2}}{3}$ (D) $\frac{8\sqrt{2}-11}{3}$
(E) $\frac{6-4\sqrt{2}}{3}$

2007 AMC 12 A, Problem #20—

"Removing the corners removes two segments of equal length x from each edge of the cube.Each removed corner is a tetrahedron whose altitude is x and whose base is an isosceles right triangle with leg length x."

Solution

Answer (B): Removing the corners removes two segments of equal length from each edge of the cube. Call that length x. Then each octagon has side length $\sqrt{2}x$, and the cube has edge length $1 = \left(2+\sqrt{2}\right)x$, so

$$x = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}.$$

Each removed corner is a tetrahedron whose altitude is x and whose base is an isosceles right triangle with leg length x. Thus the total volume of the eight tetrahedra is

$$8 \cdot \frac{1}{3} \cdot x \cdot \frac{1}{2}x^2 = \frac{1}{6}\left(2-\sqrt{2}\right)^3 = \frac{10-7\sqrt{2}}{3}.$$

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.
Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Tetrahedra

hb07-20

The parallelogram bounded by the lines $y = ax + c$, $y = ax + d$, $y = bx + c$, and $y = bx + d$ has area 18. The parallelogram bounded by the lines $y = ax + c$, $y = ax - d$, $y = bx + c$, and $y = bx - d$ has area 72. Given that a, b, c, and d are positive integers, what is the smallest possible value of $a + b + c + d$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

2007 AMC 12 B, Problem #20—
"Each parallelogram is composed of two triangles."

Solution

Answer (D): Two vertices of the first parallelogram are at $(0, c)$ and $(0, d)$. The x-coordinates of the other two vertices satisfy $ax + c = bx + d$ and $ax + d = bx + c$, so the x-coordinates are $\pm(c - d)/(b - a)$. Thus the parallelogram is composed of two triangles, each of which has area

$$9 = \frac{1}{2} \cdot |c - d| \cdot \left| \frac{c - d}{b - a} \right|.$$

It follows that $(c - d)^2 = 18|b - a|$. By a similar argument using the second parallelogram, $(c + d)^2 = 72|b - a|$. Subtracting the first equation from the second yields $4cd = 54|b - a|$, so $2cd = 27|b - a|$. Thus $|b - a|$ is even, and $a + b$ is minimized when $\{a, b\} = \{1, 3\}$. Also, cd is a multiple of 27, and $c + d$ is minimized when $\{c, d\} = \{3, 9\}$. Hence the smallest possible value of $a + b + c + d$ is $1 + 3 + 3 + 9 = 16$. Note that the required conditions are satisfied when $(a, b, c, d) = (1, 3, 3, 9)$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals > Parallelogram

tb07-20

A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

(A) 100 **(B)** 125 **(C)** 600 **(D)** 2300 **(E)** 3600

2007 AMC 10 B, Problem #20—
"After one of the 25 blocks is chosen, 16 of the remaining blocks do not share its row or column."

Solution

Answer (C): After one of the 25 blocks is chosen, 16 of the remaining blocks do not share its row or column. After the second block is chosen, 9 of the remaining blocks do not share a row or column with either of the first two. Because the three blocks can be chosen in any order, the number of different combinations is

$$\frac{25 \cdot 16 \cdot 9}{3!} = 25 \cdot 8 \cdot 3 = 600.$$

Difficulty: Hard

NCTM Standard: Geometry Standard for Grades 9-12: Use visualization, spatial reasoning, and geometric modeling to solve problems.

Mathworld.com Classification: Geometry > Plane Geometry > Squares

ha07-21

The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

(A) the coefficient of x^2 **(B)** the coefficient of x
(C) the y-intercept of the graph of $y = f(x)$
(D) one of the x-intercepts of the graph of $y = f(x)$
(E) the mean of the x-intercepts of the graph of $y = f(x)$

2007 AMC 12 A, Problem #21— "The product of the zeros of f is c/a, and the sum of the zeros is $-b/a$."

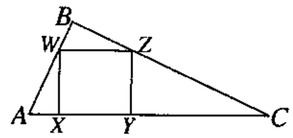
Solution

Answer (A): The product of the zeros of f is c/a, and the sum of the zeros is $-b/a$. Because these two numbers are equal, $c = -b$, and the sum of the coefficients is $a + b + c = a$, which is the coefficient of x^2. To see that none of the other choices is correct, let $f(x) = -2x^2 - 4x + 4$. The zeros of f are $-1 \pm \sqrt{3}$, so the sum of the zeros, the product of the zeros, and the sum of the coefficients are all -2. However, the coefficient of x is -4, the y-intercept is 4, the x-intercepts are $-1 \pm \sqrt{3}$, and the mean of the x-intercepts is -1.

Difficulty: Hard
NCTM Standard: Algebra Standard for Grade 9-12: analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior.
Mathworld.com Classification: Algebra > Algebraic Equations > Quadratic Equation

tb07-21

Right $\triangle ABC$ has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X and Y on $\overline{AC}$, W on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square?



(A) $\frac{3}{2}$ **(B)** $\frac{60}{37}$ **(C)** $\frac{12}{7}$ **(D)** $\frac{23}{13}$ **(E)** 2

2007 AMC 10 B, Problem #21— "$\triangle ABC$ and $\triangle WBZ$ are similar."

Solution

Answer (B): Let s be the side length of the square, and let h be the length of the altitude of $\triangle ABC$ from B. Because $\triangle ABC$ and $\triangle WBZ$ are similar, it follows that $\frac{h-s}{s} = \frac{h}{AC} = \frac{h}{5}$, so $s = \frac{5h}{5+h}$. Because $h = 3 \cdot 4/5 = 12/5$, the side length of the square is

$$s = \frac{5(12/5)}{5 + 12/5} = \frac{60}{37}.$$

OR

Because $\triangle WBZ$ is similar to $\triangle ABC$, we have $BZ = \frac{4}{5}s$ and $CZ = 4 - \frac{4}{5}s$.

Because $\triangle ZYC$ is similar to $\triangle ABC$, we have $\frac{s}{4-(4/5)s} = \frac{3}{5}$. Thus

$$5s = 12 - \frac{12}{5}s \quad \text{and} \quad s = \frac{60}{37}.$$

Difficulty: Hard

NCTM Standard: Geometry Standard for Grades 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Similar Triangles
Geometry > Plane Geometry > Squares

hb07-22

Two particles move along the edges of equilateral $\triangle ABC$ in the direction

$$A \rightarrow B \rightarrow C \rightarrow A,$$

starting simultaneously and moving at the same speed. One starts at A, and the other starts at the midpoint of $\overline{BC}$. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of $\triangle ABC$?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

2007 AMC 12 B, Problem #22—
"Imagine a third particle that moves in such a way that it is always halfway between the first two."

Solution

Answer (A): Imagine a third particle that moves in such a way that it is always halfway between the first two. Let D, E, and F denote the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let X, Y, and Z denote the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively. When the first particle is at A, the second is at D and the third is at X. When the first particle is at F, the second is at C and the third is at Z. Between those two instants, both coordinates of the first two particles are linear functions of time. Because the average of two linear functions is linear, the third particle traverses $\overline{XZ}$. Similarly, the third particle traverses $\overline{ZY}$ as the first traverses $\overline{FB}$ and the second traverses $\overline{CE}$. Finally, as the first particle traverses $\overline{BD}$ and the second traverses $\overline{EA}$, the third traverses $\overline{YX}$. As the first two particles return to A and D, respectively, the third makes a second circuit of $\triangle XYZ$.
If O is the center of $\triangle ABC$, then by symmetry O is also the center of equilateral $\triangle XYZ$. Note that $OZ = OC - ZC = \frac{2}{3}CF - \frac{1}{2}CF = \frac{1}{6}CF$, so the ratio of the area of $\triangle XYZ$ to that of $\triangle ABC$ is

$$\left(\frac{OZ}{OC}\right)^2 = \left(\frac{\frac{1}{6}CF}{\frac{2}{3}CF}\right)^2 = \frac{1}{16}.$$

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Equilateral Triangles

tb07-22

A player chooses one of the numbers 1 through 4. After the choice has been made, two regular four-sided (tetrahedral) dice are rolled, with the sides of the dice numbered 1 through 4. If the number chosen appears on the bottom of exactly one die after it is rolled, then the player wins \$1. If the number chosen appears on the bottom of both of the dice, then the player wins \$2. If the number chosen does not appear on the bottom of either of the dice, the player loses \$1. What is the expected return to the player, in dollars, for one roll of the dice?

(A) $-\frac{1}{8}$ **(B)** $-\frac{1}{16}$ **(C)** 0 **(D)** $\frac{1}{16}$ **(E)** $\frac{1}{8}$

2007 AMC 10 B, Problem #22—
"Find out the probability of the number appearing 0, 1, and 2 times."

Solution

Answer (B): The probability of the number appearing 0, 1, and 2 times is

$$P(0) = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}, \quad P(1) = 2\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{6}{16}, \quad \text{and} \quad P(2) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16},$$

respectively. So the expected return, in dollars, to the player is

$$P(0)\cdot(-1) + P(1)\cdot(1) + P(2)\cdot(2) = \frac{-9+6+2}{16} = -\frac{1}{16}.$$

Difficulty: Hard
NCTM Standard: Data Analysis and Probability for Grades 9-12: Understand and apply basic concepts of probability.
Mathworld.com Classification: Probability and Statistics > Probability

ha07-23

Square $ABCD$ has area 36, and $\overline{AB}$ is parallel to the x-axis. Vertices A, B, and C are on the graphs of $y = \log_a x$, $y = 2\log_a x$, and $y = 3\log_a x$, respectively. What is a?

(A) $\sqrt[6]{3}$ **(B)** $\sqrt{3}$ **(C)** $\sqrt[3]{6}$ **(D)** $\sqrt{6}$ **(E)** 6

2007 AMC 12 A, Problem #23—

"Since $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q$ for some p and q."

Solution

Answer (A): Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Then $AB = 6 = |p - q|$. Because $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$. Thus $|q^2 - q| = 6$, and the only positive solution is $q = 3$. Note that $C = (q, 3\log_a q)$, so $BC = 6 = \log_a q$, from which $a^6 = q = 3$ and $a = \sqrt[6]{3}$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: analyze properties and determine attributes of two- and three-dimensional objects.
Algebra Standard for Grade 9-12: use symbolic algebra to represent and explain mathematical relationships.
Mathworld.com Classification: Calculus and Analysis > Special Functions > Logarithms

hb07-23

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

2007 AMC 12 B, Problem #23—
"Set up an equation to represent the relationship of area and perimeter."

Solution

Answer (A): Let the triangle have leg lengths a and b, with $a \le b$. The given condition implies that

$$\frac{1}{2}ab = 3\left(a + b + \sqrt{a^2 + b^2}\right),$$

so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring both sides and simplifying yields

$$ab\,(ab - 12a - 12b + 72) = 0,$$

from which

$$(a - 12)(b - 12) = 72.$$

The positive integer solutions of the last equation are $(a, b) = (3, 4)$, $(13, 84)$, $(14, 48)$, $(15, 36)$, $(16, 30)$, $(18, 24)$, and $(20, 21)$. However, the solution $(3, 4)$ is extraneous, and there are six right triangles with the required property.

Difficulty: Hard

NCTM Standard: Geometry Standard for Grade 9-12: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Right Triangle

tb07-23

A pyramid with a square base is cut by a plane that is parallel to its base and is 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

(A) 2 **(B)** $2+\sqrt{2}$ **(C)** $1+2\sqrt{2}$ **(D)** 4 **(E)** $4+2\sqrt{2}$

2007 AMC 10 B, Problem #23—

"Because the two pyramids are similar, the ratio of their altitudes is the square root of the ratio of their surface areas."

Solution

Answer (E): Let h be the altitude of the original pyramid. Then the altitude of the smaller pyramid is $h-2$. Because the two pyramids are similar, the ratio of their altitudes is the square root of the ratio of their surface areas. Thus $h/(h-2)=\sqrt{2}$, so

$$h=\frac{2\sqrt{2}}{\sqrt{2}-1}=4+2\sqrt{2}.$$

Difficulty: Hard

NCTM Standard: Geometry for Grades 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Pyramids

ha07-24

For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

(A) 2,014,524 **(B)** 2,015,028 **(C)** 2,015,033 **(D)** 2,016,532
(E) 2,017,033

2007 AMC 12 A, Problem #24—

"$F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$."

Solution

Answer (D): Note that $F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$. For each n, $\sin nx \geq 0$ on each interval $[(2k-2)\pi/n, (2k-1)\pi/n]$ where k is a positive integer and $2k - 1 \leq n$. The number of such intervals is $n/2$ if n is even and $(n+1)/2$ if n is odd. The graphs intersect twice on each interval unless $\sin x = 1 = \sin nx$ at some point in the interval, in which case the graphs intersect once. This last equation is satisfied if and only if $n \equiv 1 \pmod 4$ and the interval contains $\pi/2$. If n is even, this count does not include the point of intersection at $(\pi, 0)$. Therefore $F(n) = 2(n/2) + 1 = n + 1$ if n is even, $F(n) = 2(n+1)/2 = n + 1$ if $n \equiv 3 \pmod 4$, and $F(n) = n$ if $n \equiv 1 \pmod 4$. Hence

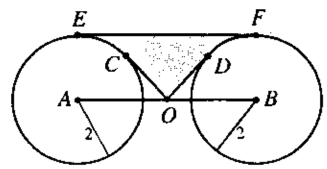
$$\sum_{n=2}^{2007} F(n) = \left(\sum_{n=2}^{2007} (n+1)\right) - \left\lfloor \frac{2007-1}{4} \right\rfloor = \frac{(2006)(3+2008)}{2} - 501 = 2{,}016{,}532.$$

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: analyze properties and determine attributes of two- and three-dimensional objects.

Mathworld.com Classification: Geometry > Trigonometry > General Trigonometry
Algebra > Sums

ta07-24

Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and $\overline{EF}$ is a common tangent. What is the area of the shaded region $ECODF$?

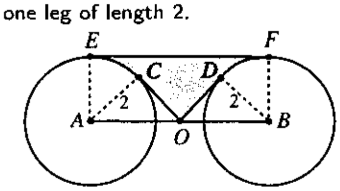


(A) $\frac{8\sqrt{2}}{3}$ **(B)** $8\sqrt{2} - 4 - \pi$ **(C)** $4\sqrt{2}$ **(D)** $4\sqrt{2} + \frac{\pi}{8}$ **(E)** $8\sqrt{2} - 2 - \frac{\pi}{2}$

2007 AMC 10 A, Problem #24— "area of ECODF = Rectangle ABFE - (triangle ACO + triangle BDO) - (sector CAE + sector DBF) "

Solution

Answer (B): Rectangle $ABFE$ has area $AE \cdot AB = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$. Right triangles ACO and BDO each have hypotenuse $2\sqrt{2}$ and one leg of length 2.



Hence they are each isosceles, and each has area $(1/2)\left(2^2\right) = 2$. Angles CAE and DBF are each $45°$, so sectors CAE and DBF each have area

$$\frac{1}{8} \cdot \pi \cdot 2^2 = \frac{\pi}{2}.$$

Thus the area of the shaded region is

$$8\sqrt{2} - 2 \cdot 2 - 2 \cdot \frac{\pi}{2} = 8\sqrt{2} - 4 - \pi.$$

Difficulty: Hard
NCTM Standard: Geometry Standard: apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Plane Geometry > Miscellaneous Plane Geometry > Area

tb07-24

Let n denote the smallest positive integer that is divisible by both 4 and 9, and whose base-10 representation consists of only 4's and 9's, with at least one of each. What are the last four digits of n?

(A) 4444 **(B)** 4494 **(C)** 4944 **(D)** 9444 **(E)** 9944

2007 AMC 10 B, Problem #24—

"The sum of the digits of n must be a multiple of 9, and the last two digits of n must each be 4."

Solution

Answer (C): Since n is divisible by 9, the sum of the digits of n must be a multiple of 9. At least one digit of n is 4, so at least nine digits must be 4, and at least one digit must be 9. For n to be divisible by 4, the last two digits of n must each be 4. These conditions are satisfied by several ten-digit numbers, of which the smallest is 4,444,444,944.

Difficulty: Hard

NCTM Standard: Number and Operations for Grades 9-12: Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

Mathworld.com Classification: Number Theory > Divisors > Divisible

Number Theory > Arithmetic > Number Bases > Base

hb07-25

Points A,B,C,D, and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90°$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\sqrt{5}$ **(E)** $\sqrt{6}$

2007 AMC 12 B, Problem #25—
"Introduce a coordinate system in which $D = (-1, 0, 0)$, $E = (1, 0, 0)$, and $\triangle ABC$ lies in a plane $z = k > 0$."

Solution

Answer (C): Introduce a coordinate system in which $D = (-1, 0, 0)$, $E = (1, 0, 0)$, and $\triangle ABC$ lies in a plane $z = k > 0$. Because $\angle CDE$ and $\angle DEA$ are right angles, A and C are located on circles of radius 2 centered at E and D in the planes $x = 1$ and $x = -1$, respectively. Thus $A = (1, y_1, k)$ and $C = (-1, y_2, k)$, where $y_j = \pm\sqrt{4 - k^2}$ for $j = 1$ and 2. Because $AC = 2\sqrt{2}$, it follows that $(1-(-1))^2+(y_1 - y_2)^2 = \left(2\sqrt{2}\right)^2$. If $y_1 = y_2$, there is no solution, so $y_1 = -y_2$. It may be assumed without loss of generality that $y_1 > 0$, in which case $y_1 = 1$ and $y_2 = -1$. It follows that $k = \sqrt{3}$, so $A = (1, 1, \sqrt{3})$, $C = \left(-1, -1, \sqrt{3}\right)$, and B is one of the points $\left(1, -1, \sqrt{3}\right)$ or $\left(-1, 1, \sqrt{3}\right)$. In the first case, $BE = 2$ and $\overline{BE} \perp \overline{DE}$. In the second case, $BD = 2$ and $\overline{BD} \perp \overline{DE}$. In either case, the area of $\triangle BDE$ is $(1/2)(2)(2) = 2$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grade 9-12: Specify locations and describe spatial relationships using coordinate geometry and other representational systems.
Mathworld.com Classification: Geometry > Surfaces > Planes
Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Right Triangle