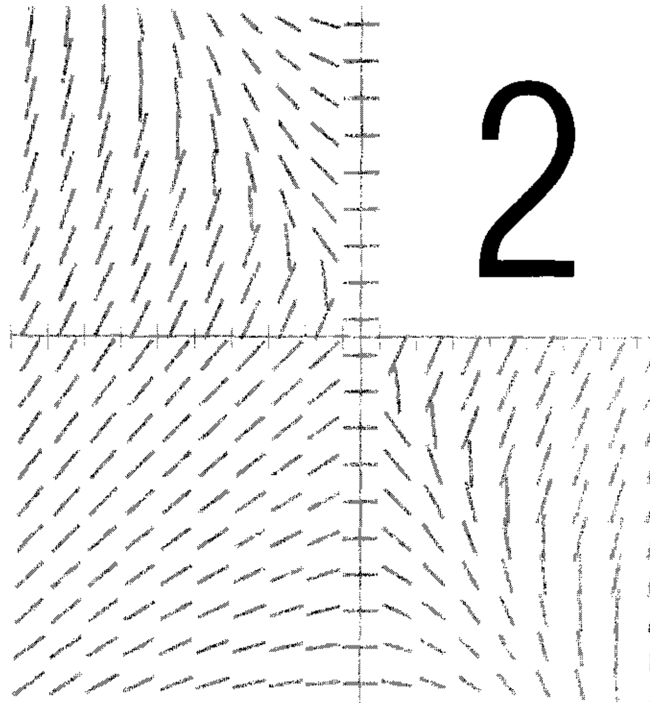
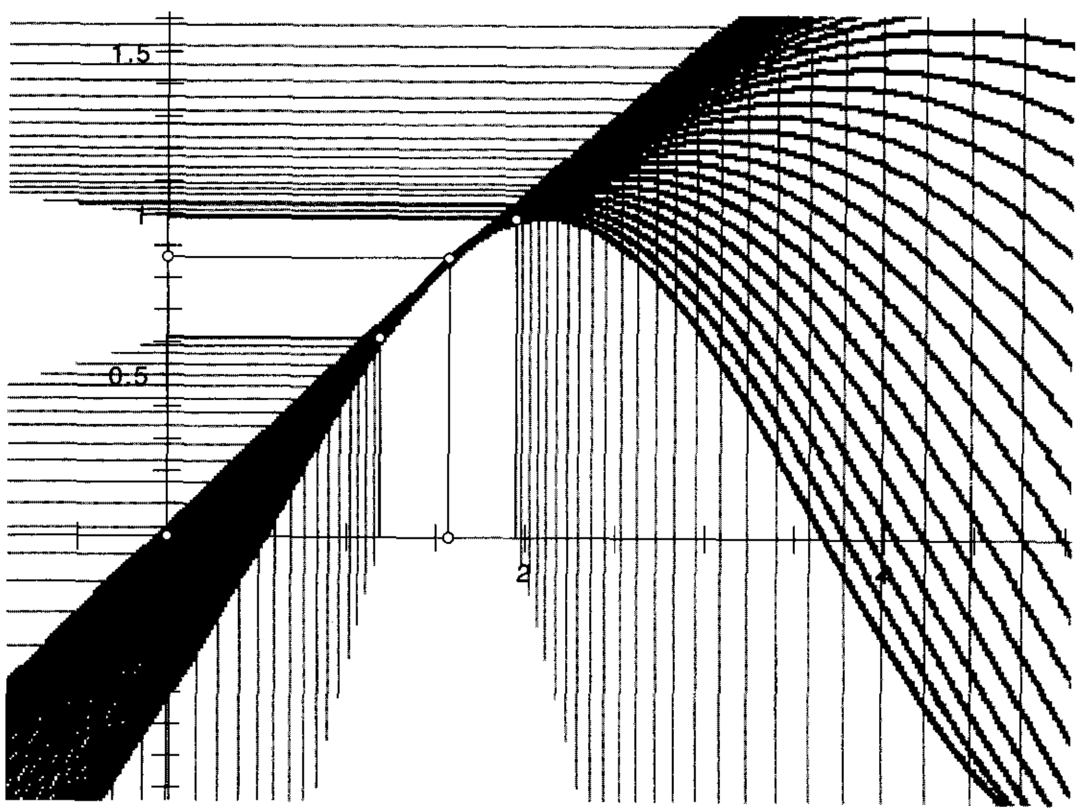
2



Exploring Limits



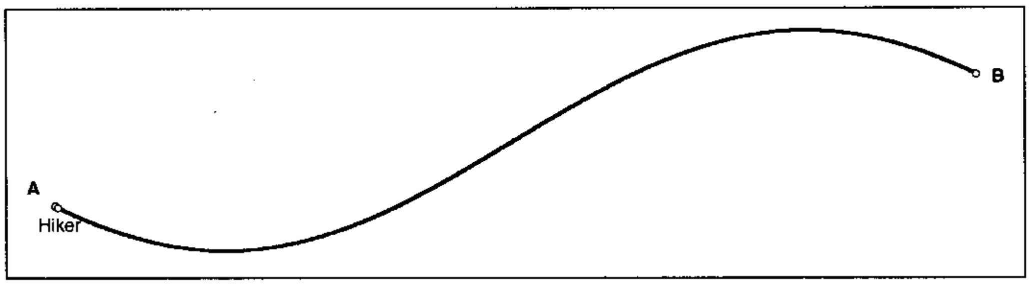
Continuous or Discontinuous? Name(s): ____________

As you've studied functions, you've learned about the different kinds of behavior functions can have. Some functions are constant, some are increasing, and some are decreasing. Some functions are linear and some are nonlinear. Some functions are defined on an unlimited domain, and others (such as the square root function) have a limited domain.

In this activity you'll look at a different aspect of the behavior of a function, an aspect which is particularly important in calculus: whether a function is continuous or discontinuous.

Sketch and Investigate

Imagine that you are a mountain climber making your way up Mt. Everest.



1. **Open** the sketch **Continuity.gsp** in the **Exploring Limits** folder. This sketch shows a rough profile of a glacier crossing, but without enough detail to see every little bump and dip.

This is not a particularly steep portion of your climb. Do you expect to have any difficulty hiking from point A to point B?

2. Press the *Start Hiking* button to begin the hike across the glacier. Observe what happens as the hiker heads for point B from point A. Press the *Show Details* and *Hide Rough Profile* buttons to show why you were rudely surprised.

Just as you must be vigilant about discontinuities in the surface of a glacier when you're hiking, you must also be vigilant about discontinuities in functions when you're doing calculus. You'll explore three different types of discontinuities in this activity.

Click on the page tabs in the bottom-left corner of the window to switch between pages.

3. Go to page 2 to explore the first type of discontinuity.

Point P was constructed on a discontinuous function. By slowly moving point P, you can trace out the function and find the discontinuity. If you drag point P more slowly, your trace will be more accurate.

4. Move point P to trace the function and try to locate the discontinuity.

Q1 At what x-value does the discontinuity occur?

5. Select point P, and then choose a different color from the Color submenu of the Display menu.

6. Press the *Trace Function* button to move point P automatically. Trace out one complete cycle, and then press the button again to stop the trace. Compare this trace with yours. Did you get the entire function, or did you miss part of it?

Pressing the Esc key once also stops the animation.

7. Press the Esc key several times to clear your traces.

Let's explore the function's behavior near the discontinuity in detail. It will be useful to do this in two parts, exploring the behavior both to the left and to the right of the discontinuity.

8. Press the *Show Left Branch* button. This will allow you to explore the behavior to the left of the discontinuity.

9. Two points appear: point A on the x-axis and the corresponding point $(x_A, f(x_A))$ on the function f—labeled point Q. Drag point A left and right. See how far to the right you can drag it.

Q2 What happens to point Q when you drag point A all the way to the right?

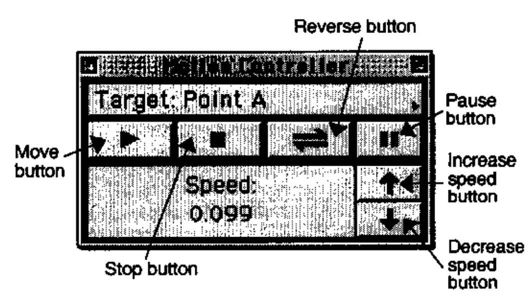
Click once in an empty part of the sketch to deselect an object.

10. Move point A back to the left of the discontinuity so that point Q is showing, then deselect point A. Select point Q, then measure its coordinates by choosing **Coordinates** from the Measure menu.

Q3 What happens to the coordinates of point Q when you drag point A as far right as it can go?

You can use animation to move point Q very close to the discontinuity—if you move point A very slowly.

11. To set this up: Select point A and choose **Animate Point** from the Display menu. Use the Motion Controller to decrease the speed to approximately 0.01. Then press the *Pause* button in the Motion Controller, and drag point A near the discontinuity.



If point A is moving away from the discontinuity, press the *Reverse* button to change its direction.

12. Release the *Pause* button so that point A starts moving again, very slowly. Pause the motion again when point Q is very close to the discontinuity. By using the *Reverse* and *Pause* buttons and adjusting the speed you should be able to move point Q very close to the discontinuity.

Q4 Write down the coordinates of the closest point you achieved.

Q5 What do you think is the left-hand limit of the y-coordinate of point Q as the x-coordinate approaches 2?

Now let's explore the other side.

13. Hide the left branch and show the right branch. Two points appear: point B on the x-axis and the corresponding point $(x_B, f(x_B))$ on the function f—labeled point R. Measure point R's coordinates.

You'll create two parameters and a *Move* button to be more systematic about moving point R close to the discontinuity.

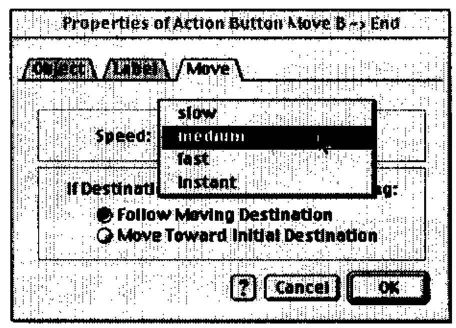
To enter a subscript type it in brackets. For End_x you would type End[x].

14. Choose **New Parameter** from the Graph menu. Name the first parameter End_x, give it a value of 2.1, and then click OK.

15. Make a second parameter with the name of End_y and a value of 0.

16. Select the measurements End_x and End_y, in that order. Then choose **Plot As (x, y)** from the Graph menu. Label this point *End*.

17. To create a button that will move point B to point *End*, with the **Arrow** tool, select point B and point *End*, in that order. Then choose **Edit | Action Buttons | Movement.** On the Move panel, choose **medium** speed and click OK.



Make sure the Motion Controller is not on pause, otherwise nothing can move.

18. Click once in an empty spot to deselect the *Move* button. Then select point *End* and choose **Display | Hide Plotted Point** to hide it. Then press the *Move B→End* button.

Q6 What are the coordinates of point R?

19. Double-click on the measurement End_x. Change its value to 2.01. Then press the *Move B→End* button.

Q7 What are the coordinates of point R?

Repeat step 19, putting point *End* even closer, at (2.001, 0).

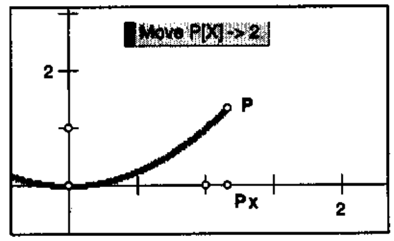
Q8 What are the coordinates of point R this time?

Q9 What is the right-hand limit of the y-coordinate of point R as the x-coordinate approaches 2? Is this the value that you found for the left branch? Does the limit exist at $x = 2$?

20. Go to page 3 to see a different type of discontinuity. Trace this function by dragging point P_x (*not* point P) or by pressing the *Trace Function* button. Do you see a discontinuity as you trace?

21. Clear your traces and show the function plot. Can you see a discontinuity now?

There is a discontinuity here, but it's even narrower than the crevasse that the hiker fell into.



22. Press the *Move* $P_x \rightarrow 2$ button. Can you see a discontinuity now?

Q10 What happens to point P when point P_x is at exactly $x = 2$?

23. Hide the function plot and erase traces. Then show the left branch, measure the x- and y-coordinates of point Q, and determine the limit of the y-value as the x-value approaches 2 from the left.

24. Hide the left branch, show the right branch, measure the coordinates of point R, and determine the limit of the y-value as the x-value approaches 2 from the right.

Q11 What is the limit of the y-value as x approaches 2 from the left? From the right? What is the limit as x approaches 2?

25. Go to page 4 to see the last type of discontinuity. Trace this function by dragging point P or by selecting point P and choosing **Animate** from the Display menu. Find the discontinuity as you trace.

26. Show the function plot and use point A to find three x- and y-values just to the left of the discontinuity.

Q12 List the three closest points you found in step 26 in numerical order.

27. Find three x- and y-values just to the right of the discontinuity.

Q13 List the three closest points you found in step 27 in numerical order.

Q14 What is the limit of the y-value as x approaches –1 from the left? From the right?

On pages 2, 3, and 4 you explored three different types of discontinuities in functions. Answer the following questions about these discontinuities. Use a separate sheet of paper, and write in complete sentences.

Q15 Give a name to each type of discontinuity. First come up with a descriptive name of your own, and then do some research to find out what names other people have used.

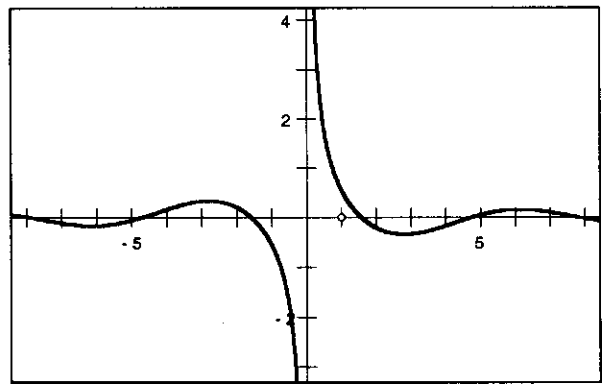
Q16 What are the similar features of all three types of discontinuities?

Q17 For each type of discontinuity, describe the features that make it different from the other two.

Q18 Which type of mathematical discontinuity best describes the situation of the hiker who fell into the crevasse? Justify your answer!

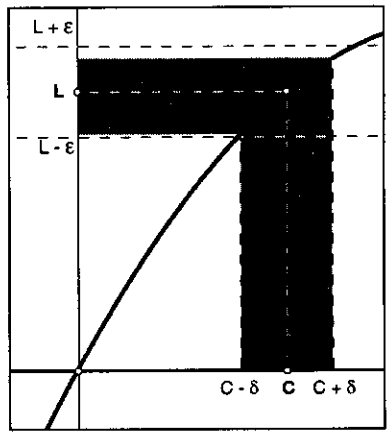
Explore More

Use Sketchpad to graph each function listed below. For each one, locate and explore its discontinuity and decide which type of discontinuity it is. (Some functions may have more than one discontinuity.)



1. $f(x) = \text{trunc}(x)$
2. $f(x) = \left|\dfrac{1}{x-1}\right|$
3. $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$
4. $f(x) = \dfrac{\sin(x)}{x}$
5. $f(x) = \dfrac{\cos(x)}{x}$
6. $f(x) = \tan(x)$
7. $f(x) = \dfrac{x^2 - 3x + 2}{|x - 2|}$

Delta, Epsilon, and Limits Name(s): ______________



You know that a function $f(x)$ has a limit, L, at $x = C$ if you can make the y-coordinate of the function as close as you want to L by choosing an x-coordinate really close to $x = C$.

More formally, L is the limit of $f(x)$ as x approaches C if and only if, for any positive number epsilon, ε, there is a positive number delta, δ, such that if x is within δ units of C ($x \neq C$), the y-value of f is within ε units of L.

In this activity, you'll explore this definition visually. For a given function and for given values of C, L, and ε, you'll adjust the value of δ to test the definition.

Sketch and Investigate

The constants a, b, and c are controlled by sliders. You can change the function by changing the values of the constants a, b, and c.

1. **Open** the sketch **EpsilonDelta.gsp** in the **Exploring Limits** folder. This sketch shows the graph of $f(x) = ax^2 + bx + c$.

2. Experiment with the *epsilon* slider, which determines an interval on the y-axis about L. This interval is represented by a vertical red segment containing all y-values within ε of L.

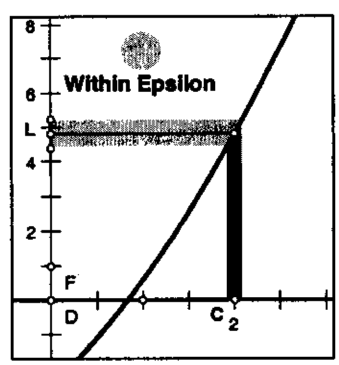
3. Experiment with the *delta* slider, which determines an interval on the x-axis about C. This interval is represented by the width of a vertical blue polygon and contains all x-values within δ of C.

The intersection of the left and right edges of this blue polygon with the function determines another polygon (horizontal this time). The height of this second green polygon represents the y-values of the function when the x-value is restricted to within δ of C.

You can reset these constraints by pressing the *Show Parameters* button. Double-click on a, b, or c to change its value.

4. Set the sliders for a, b, and c to $a = 0.6$, $b = 2.5$, and $c = -2$ by pressing the *Set Quadratic* button.

5. Press the *Case 1* button to set the values of C, L, and ε.



The definition of a limit will be satisfied graphically if you can adjust δ—using the *delta* slider—in such a way that all the possible y-values represented by the horizontal green polygon fall within this red interval (that is, they fall within ε of L or in the interval $[L - \varepsilon, L + \varepsilon]$).

Use keyboard arrows to move one pixel at a time.

6. Adjust the *delta* slider until the horizontal green polygon falls *just* within the red $[L - \varepsilon, L + \varepsilon]$ interval on the y-axis. (The circular indicator changes from red to green when *all* possible y-values lie within the interval.)

Q1 What is the largest value you found for δ for this value of ε? Record the values of C, L, ε, and δ in a table on a separate sheet of paper.

7. Move the *epsilon* slider to set ε to roughly half its previous value. Adjust δ to work with this new value of ε. Record this new set of results in your table.

Q2 When you set ε to half its previous value, did your value for δ reduce by half as well? Try other fractions of ε, such as 2/3 or 2/5.

Drag the unit points to resize your window and then press the *Case* button again to reset the value of ε.

8. Press the *Case* 2 button, and repeat steps 6–7 and Q1–Q2. Add your results to the table you made in Q1.

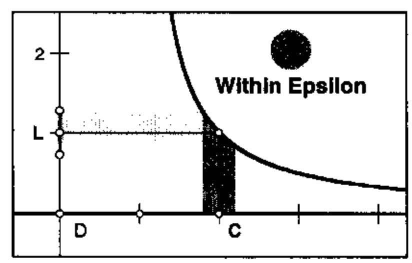
9. Repeat for Cases 3 and 4, adding your results to the table each time.

Q3 Was it possible in each case to set a value of δ that satisfied the definition of a limit? If not, note the values for C, L, and ε for which it was not possible. Did your effort fail because L does not exist, or because you could not adjust δ correctly? What can you conclude about the function as x approaches C in each case?

Q4 Do you think it would be possible to find such a value of δ graphically for every possible value of C and ε? If not, what values of C and ε would not work?

You will need to resize your window for some of these cases. Don't forget to press the *Case* button after resizing to reset the value for ε.

10. Go to page 2 of the document and follow the same steps described above. For each case, adjust the *delta* slider so that the resulting y-values fall within ε of the limit L, and record your results in a table. Record just one set of results for each case.



Q5 Was it possible in each case to set a value of δ that satisfies the definition of a limit? If not, note the values for C, L, and ε for which it was not possible. Did your effort fail because L does not exist, or because you could not adjust δ correctly? What can you conclude about the function as x approaches C?

Explore More

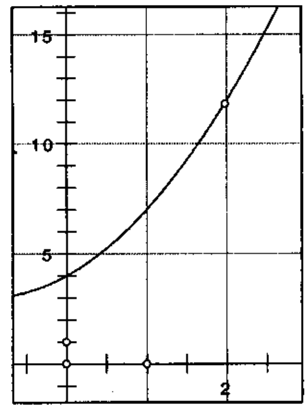
To graph a new function, double-click on the expression for $f(x)$, delete the current expression, and enter your own.

Go to page 3 of the document and experiment with the fourth-degree polynomial there or graph a new function.

Go to page 4 of the document and experiment with the trigonometric function there or graph a new function.

How Close Do You Go?

Name(s): ____________________



You have seen that the y-value of a function can approach a limiting value as x approaches a constant c even if f is undefined. For example, if you plot the function $q(x) = \frac{x^3 - 8}{x - 2}$, you get a parabola with a gap or hole at $x = 2$. But as x approaches 2, the y-value of the function approaches 12.

You have learned that L is the limit of a function f as x approaches c if you can make the y-value of the function as close as you want to L by making x sufficiently close to c. In this activity, you will explore this definition graphically for various functions.

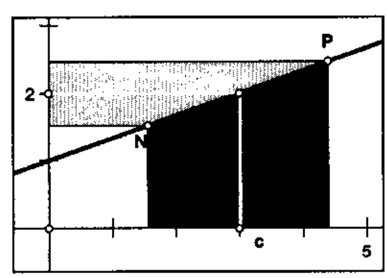
Sketch and Investigate

1. **Open** the document **Limits2.gsp** in the **Exploring Limits** folder.

On page 1, you have the plot of the function $f(x) = \frac{x^2 - 9}{3x - 9}$, which has a hole at $x = 3$. Points P and N will allow you to evaluate the function for x-values near $c = 3$.

Q1 Try to find $f(3)$ by direct substitution. What happens?

Q2 What limiting value does the plot suggest as x approaches 3? Check your answer by simplifying the expression for f and substituting $x = 3$. Do you get the same value for the limit?



In order to show graphically that the definition holds, you need to show that you can make $f(x)$ as close as you wish to $L = 2$ by making x sufficiently close to $c = 3$.

Q3 Suppose someone tells you to keep $f(x)$ within 0.5 units of the limit 2. How close must x be to $c = 3$ to make the y-values of the function fall within that interval?

Check that $a = 3$ and $b = 2$. If they don't, edit them by double-clicking on their measurements.

To get a little closer you can zoom in around this point even though $f(x)$ is not defined at $x = 3$. The x- and *y-scale* sliders zoom the coordinate grid around the point (a, b) for the values of a and b defined in the sketch.

2. Drag points P and N as close as you can to the limit.

If you want to use both sliders at once, select the endpoint of each slider, then select either point and drag.

3. Zoom in using the sliders until you lose your axes. Then press the *Show Zoom Axes* button, and then the *Hide Main Axes* button. Zoom in again until the x-axis goes from about 2.2 to 3.8 and the y-axis from about 1.5 to 2.5.

This will be a rough estimate because you are using the tick marks on the zoom axes and eyeballing it.

4. Adjust the location of points P and N to find out how close you have to make x to 3 so that $f(x)$ is within 0.1 of the limit 2.
5. Measure the coordinates of points P and N to make sure that all of the horizontal rectangle is within 0.1 of the limit.

Q4 How close do you have to make x to 3 so that $f(x)$ is within 0.1 of the limit 2?

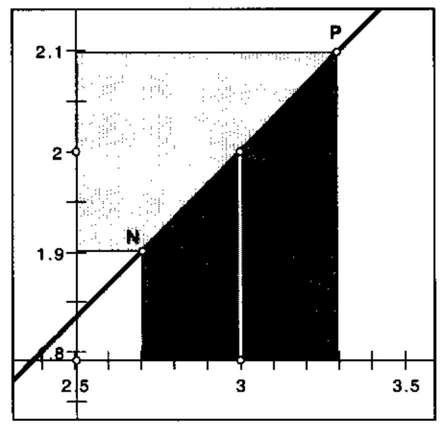
Drag points P and N as close as you can to the limit before you zoom in.

6. Adjust the sliders and the location of points P and N to show how close you must make x to 3 so that $f(x)$ is within 0.05 of 2.

Q5 How close do you have to make x to 3 so that $f(x)$ is within 0.05 of 2?

You can calculate $f(x_P) - L$ and $L - f(x_N)$ to check these limits as well.

Q6 How close do you have to make x to 3 so that $f(x)$ is within 0.01 of 2? Within 0.001 of 2?



As you saw above, to demonstrate graphically that a limit exists, you show that you can make x close enough to c so that the value of $f(x)$ will always lie within a narrow band around the limit. In the last activity you learned that this distance from L is called *epsilon*, or ε, and the distance from c is called *delta*, or δ. In the example shown here, $\varepsilon \approx 0.1$ and $\delta \approx 0.3$.

Q7 Look back at your answers to the previous questions. Does there appear to be a relationship between the value of ε and the value of δ? If so, what is this relationship?

Q8 If you look at the sketch above, the interval around $c = 3$ is symmetric going from about $3 - 0.3$ to about $3 + 0.3$. Why do you think this was true for all of your intervals above?

Let's try another function to see if we usually get a symmetric interval around our target value $x = c$.

7. Go to page 2 of the document. Here $f(x) = 2 - 4^{x-1}$ and the limit as x approaches 0.5 is 1.5. Zoom in and adjust point P and point N so that the y-values are within 0.1 of the limit.

Q9 How close do you need to make x to 0.5 on the left side of point C? On the right side of point C? What would your δ be in this case?

Now we have the hypothesis that for exponential functions (without a discontinuity) the intervals won't be symmetric. But this is just one example of a function—let's look at one more polynomial.

With the Calculator open, click on the expression for $f(x)$ and then on measurement c.

8. Go to page 3 of the document. Choose **Calculate** from the Measure menu and calculate $f(3)$.

9. Choose the measurements for c and $f(c)$ in that order and choose **Plot As (x, y)** from the Graph menu.

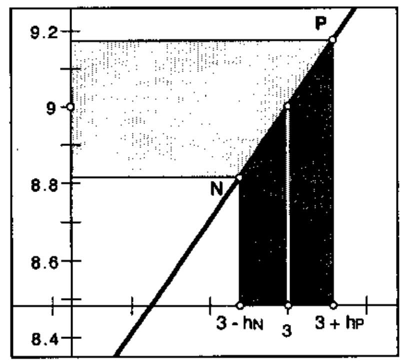
Q10 Is the function f continuous at $x = 3$?

Q11 What value does the plot suggest for the limit as x approaches 3?

10. On this page, two sliders control the location of points P and N. The slider labeled h_N controls the distance from point N to point C on the left and the slider labeled h_P controls the distance from point P to point C on the right. Experiment with the sliders to see how point P and point N move.

Once you've zoomed in, press the *Show Zoom Axes* button and then press the *Hide Main Axes* button.

11. Zoom in and adjust each slider to see how close you need to make x to 3 in order to make $f(x)$ within 0.1 of the limit. Then adjust the sliders to see how close you must make x to 3 in order to make $f(x)$ within 0.05, 0.01, and then 0.001 of the limit. Make a table of your δ and ε values.



Q12 Were your intervals around $c = 3$ symmetric? Can you find a relationship between the values that you found?

This last example had intervals that were not symmetric. (Go back and look again if you answered yes to Q12.) It turns out that most of the time, they aren't. So how was the first function different from the other two?

Let's look at one more example.

12. Go to page 4 of the document. Choose **Calculate** from the Measure menu and calculate $f(3)$.

13. Choose the measurements for c and $f(c)$ in that order, and choose **Plot As (x, y)** from the Graph menu.

Q13 Is the function f continuous at $x = 3$? What value, if any, does the plot suggest for the limit as x approaches 3?

14. Try as before to adjust each slider to see how close you need to make x to 3 in order to make $f(x)$ within 0.1 of your prediction, or, try to show why there is no limit using 0.1 as your ε.

Q14 What happened when you tried to do step 14, and how is this function different from the previous examples?

Q15 What are the left- and right-hand limits as x approaches 3? Explain why no single number can be the limit of $f(x)$ as x approaches 3.

The four examples above covered each of the classifications below except one. Figure out which classification has not been covered yet, and match the four functions above with their classifications.

A. Defined at $x = c$, limit exists at $x = c$, continuous at $x = c$.

B. Defined at $x = c$, limit exists at $x = c$, not continuous at $x = c$.

C. Defined at $x = c$, limit does not exist at $x = c$, not continuous at $x = c$.

D. Undefined at $x = c$, limit exists at $x = c$, not continuous at $x = c$.

E. Undefined at $x = c$, limit does not exist at $x = c$, not continuous at $x = c$.

Explore More

Here is a function that satisfies the missing category above:

$$f(x) = \frac{|x-3|}{x-3} + x$$

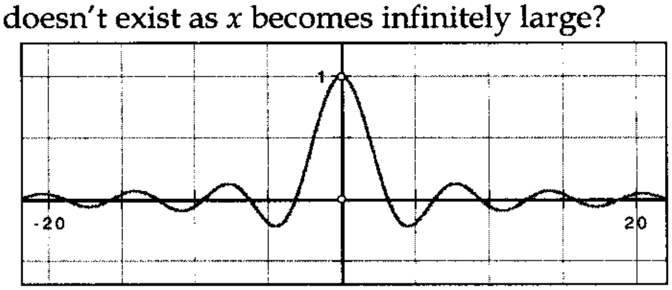
1. Go to page 5 and confirm that this function satisfies the conditions—it is undefined at $x = 3$ and the limit does not exist for $x = 3$.
2. Double-click on measurement c and change it to $c = 0$. Change b to 1.
3. Double-click on the expression for $f(x)$ and change it to $f(x) = 1 + x\cos(1/x)$. Investigate the behavior of $f(x)$ as x approaches 0, and then compare it to the function $f(x) = 1 + \cos(1/x)$.
4. Your function may get choppy as you zoom in. To fix that, select the plot of f, then choose the Plot panel from **Properties** in the Edit menu, and change the domain to a smaller interval around 0.

Q1 Does the limit as x approaches 0 exist for either of these functions?

Q2 How are these functions different in this region? How are they the same?

Q3 As x becomes infinitely large, does the y-value seem to approach a limiting value for either of these functions? If so, what value? If not, describe why not.

Q4 How could you use Sketchpad to demonstrate graphically that a limit exists or doesn't exist as x becomes infinitely large?



Slope and Limits

Name(s): ______________________

Limits are a powerful tool for describing the behavior of functions at a particular point or as x becomes infinitely large. Limits also provide the missing link we need to answer this question: What is the instantaneous rate of change of the function at a point, or what is the function's slope at any given point?

In this activity you will explore how the limit can be used to move from average rates and secant slopes to the instantaneous rate and slope at a point.

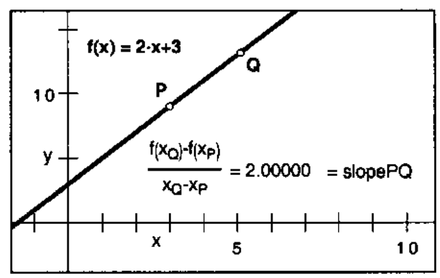
Sketch and Investigate

1. **Open** the document **SlopeandLimit.gsp** in the **Exploring Limits** folder. On the first page you will find a linear function f with its plot and points P and Q. Point P is fixed by the parameter x_P. Point Q is free to move along the function plot.

Choose **Calculate** from the Measure menu and click on each measurement to enter it into the calculator. Don't forget parentheses.

2. Find the average rate of change between points P and Q by calculating the slope of line PQ:

$$slopePQ = \frac{f(x_Q) - f(x_P)}{x_Q - x_P}$$



Label this measurement *slopePQ*.

3. Does your result make sense? Move point Q along the function plot, and confirm that the value of the calculation does not change.

The coordinates of point P are (3, 9), and because point Q is free to move along the function plot, its coordinates are $(x_Q, 2x_Q + 3)$. So the slope of the line from a fixed point P to point Q is *always:*

$$slopePQ = \frac{f(x_Q) - f(x_P)}{x_Q - x_P} = \frac{(2x_Q + 3) - 9}{x_Q - 3} = \frac{2x_Q - 6}{x_Q - 3} = \frac{2(x_Q - 3)}{x_Q - 3} = 2$$

The fact that the slope of a linear function at a point is indeed its usual slope should not have been surprising. But what is the slope at a point if the function is not linear? Is it even possible to ask such a question?

Double-click on the expression for $f(x)$ to edit the function.

4. Move point Q to the left of point P, and change $f(x)$ to x^2.

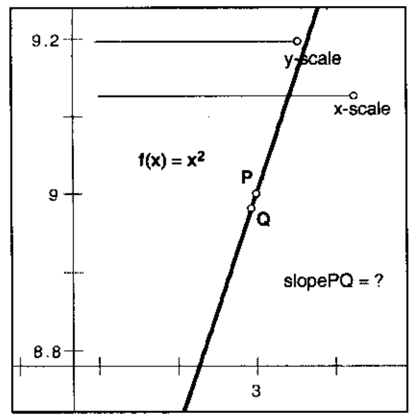
Select both points, and then choose **Line** from the Construct menu.

5. Construct a line between points P and Q, and give it a different color than the function plot by choosing **Display | Color.**

Q1 Move point Q along the function plot. Does the value of the measurement *slopePQ* stay constant as it did in step 3? If so, why do you think it stays constant? If not, describe its values as you move point Q from left to right.

Q2 What are the coordinates of point *P*? Point *Q* is not fixed, so its coordinates are not constants—what are they? Write an expression to calculate the slope of the line from point *P* to point *Q*. (See the example after step 3 if you get stuck.)

Your expression in Q2 should have been a simplified version of $\frac{f(x_Q) - f(3)}{x_Q - 3}$, or the average rate of change from point *P* ($x = 3$) to point *Q*. This expression is also a function, and it can be examined using limits.



Q3 Find $\lim_{x \to 3} \frac{f(x_Q) - f(3)}{x_Q - 3}$ algebraically.

Q4 Move point *Q* as close as you can to point *P*. Pay close attention to the measurement *slopePQ,* and look for its limiting value. As *Q* approaches *P,* does the limiting value of *slopePQ* appear to graphically support your answer to Q3?

In this document, the coordinate grid will dilate around the point $(x_P, f(x_P))$. To change the location of point *P*, edit the parameter x_P.

6. Zoom in by moving both the *x*- and *y-scale* sliders to the right. Point *Q* will appear to move away from point *P*, but note that the coordinates of point *Q* do not change.

7. Move point *Q* closer to point *P*.

Q5 What is the slope of the line *PQ* when *Q* is very close to *P*? Is this value approximately equal to the limit you calculated in Q3?

In the steps above, you examined, not the limit of the function *f* at a point, but the limit of a new function based on *f*. This new function is called the *difference quotient* at $x = 3$, but it is still the average rate of change of $f(x)$ at $x = 3$, or the slope of a secant line through point *P*.

We define *the slope of a curve at a point P* to be the limit of the slopes of these secant lines, *PQ,* as point *Q* approaches point *P*. This limit is also referred to as *the instantaneous rate of change of the function at the point P.* The limiting line itself can be used to visualize the limit.

Let's see why.

If you can zoom in at a point until the function looks linear, we say that the function is *locally linear* at that point.

8. Zoom in until the function looks like a line and you can't see any difference between the function $f(x)$ and the line *PQ*. Move point *Q* as close as possible to point *P*.

9. Choose **SlopeTool** from **Custom** tools. This tool calculates the slope between *any* two points, so use it to make two new points anywhere on the "line." (Make sure the status message is "to Point on Function Plot.")

Q6 What value did the **SlopeTool** give you for the slope of the zoomed-in plot of the function? Select either new point and drag it anywhere in the window. Does this slope value change?

Q7 Explain in your own words why this limiting line can be used to visualize the limit.

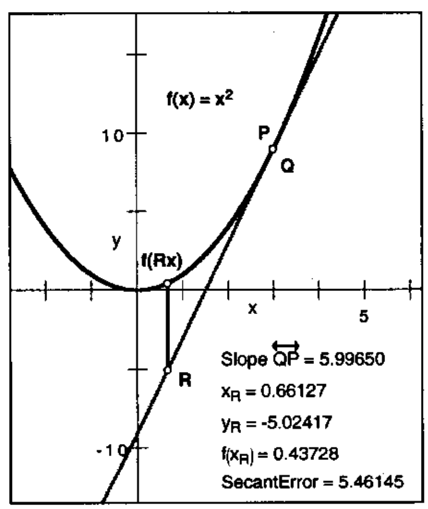
Not only are limits useful for analyzing a function's behavior, you've also seen that they give us the missing links to define an instantaneous rate of change at a point, or a function's slope at a point. Defining an instantaneous rate of change is just one reason for building a difference quotient and finding its limit, as you will see in the next activity, "Area and Limits."

Exploration 1

There is another reason why we'd like to look at a function's local linearity at a point—working with a line is usually easier than working with a general nonlinear function. If we use the line instead of the function, how much error is there? To find out, we need to look at another point close to point *P*.

If you have moved point *Q* or have changed your window since step 8, repeat step 8 above. Then you can do this step.

1. Leaving point *Q* as close as possible to point *P*, zoom back out using both sliders until the x-axis goes from about –2 to 7 and the y-axis goes from about –2 to 12.



2. Using the **Point** tool, make a new point on the line *PQ* and label it *R*.
3. Measure the x-coordinate of point *R* by selecting **Abscissa(x)** from the Measure menu. Then measure its y-coordinate (**Ordinate(y)**).
4. Calculate $f(x_R)$ and plot the point $(x_R, f(x_R))$.
5. Construct the line segment between the point plotted in step 4 and point *R*.

Q1 Explain in your own words why this last segment graphically represents the error created if you used the y-coordinate of point *R* to approximate the function's y-value, $f(x_R)$.

Q2 Drag point *R* so that $x_R \approx 2$. What would the error be if you used the y-coordinate of point *R* to approximate $f(2)$?

6. Write a formula you can use to calculate this error. Create this calculation in your sketch and label it *SecantError*.

Q3 Move point R along the line PQ and determine how close you need to get to $x = 3$ to make the error less than 0.1, 0.05, and 0.001.

Q4 You have the point P and you calculated the limiting slope for this line above. Write the equation of this limiting line in point-slope form.

7. Check your answer by choosing **Plot New Function** from the Graph menu and entering the equation for your line from Q4. Zoom in and out to see if your line matches line PQ.

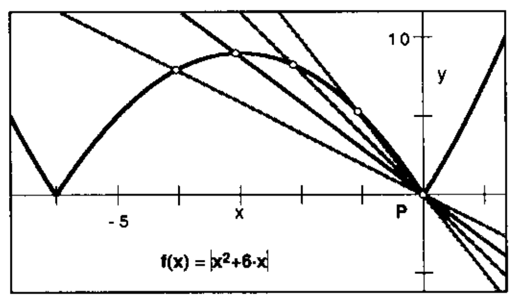
Q5 By hand (no calculators!), find the y-value for $x = 3.04$ using your line and then using your function. The calculation with the linear equation should have been much easier—what was the error?

Exploration 2

Go to page 2 to see an example of a function with points where f can't be locally linearized. Let's see what goes wrong at $x = 0$.

1. Choose the **Slope+Line** tool from **Custom** tools. This tool calculates the slope between two points and creates the secant line. Match point P (the origin) and make a new point with this tool on the function where $x \approx -4$.

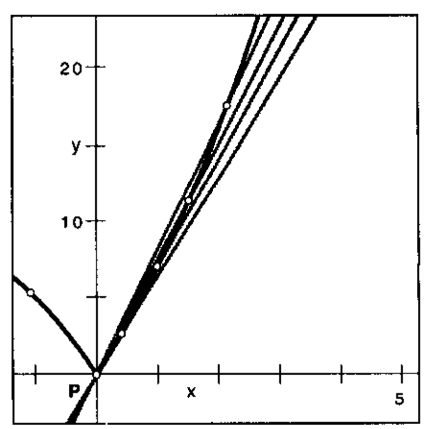
2. Use this tool to make three more secant lines from point P to $f(x)$ where $x \approx -3$, -2, and -1.



Q1 Using the four slope measurements calculated by the **Slope+Line** tool, estimate the left-hand limit for the secant slopes as x approaches 0.

3. Select each of the secant lines and choose **Display | Hide Lines.**

4. Use the **Slope+Line** tool to make secant lines from point P to $f(x)$ where $x \approx 0.5$, 1, 1.5, and 2.



Q2 Using the four slope measurements calculated by the tool, estimate the right-hand limit for the secant slopes as x approaches 0.

Q3 What is the limit of all the secant slopes as x approaches 0? Explain your answer.

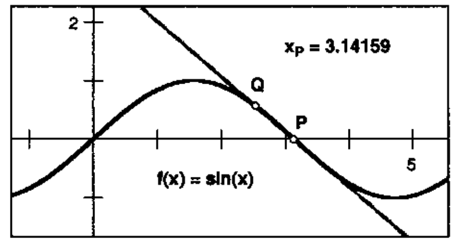
5. Pick any of the new points created by the **Slope+Line** tool, and drag it across point P. Describe what the secant line does as the point moves through P.

Q4 Explain in your own words why f is not locally linear at x.

Exploration 3

1. Go to page 3 in the document, and change the function to $f(x) = \sin(x)$, and x_P to π. (Try $x = \pi/2$ as well.)

Q1 What limit do you need to evaluate to determine the instantaneous rate of change of $\sin(x)$ at $x = \pi$? Zoom in to estimate the limit.



2. Zoom back out. Change the function to $f(x) = \ln(x)$ and x_P to 4. (Try $x = 3$ and 5 as well.)

Q2 What limit do you need to evaluate to determine the instantaneous rate of change of $\ln(x)$ at $x = 4$? Zoom in to estimate the limit.

3. Zoom back out. Change the function to $f(x) = \sqrt{x}$ and x_P to 4. (Try $x = 2$ and 5 as well.)

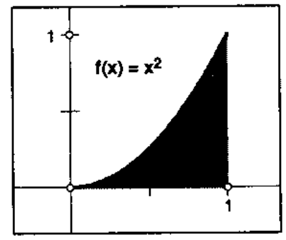
Q3 What limit do you need to evaluate to determine the instantaneous rate of change of $\sqrt{x}$ at $x = 4$? Zoom in to estimate the limit.

Q4 You may be familiar with situations where limits do not exist. Can you think of a function with a point where the instantaneous rate of change, as defined above, will not exist? Write the function and the accompanying limit, and show that this limit does not exist. (*Hint:* Think about the slope of the line you used to visualize the limit.)

Area and Limits

Name(s): ____________________

The area of the shaded region on the right looks to be a little less than 0.5 square units. But how much less? If you had to come up with one number for the most accurate measurement of the area, what would it be? How would you find that number? How might you convince someone that your approximation is the most accurate? This activity will help you begin to answer these questions.

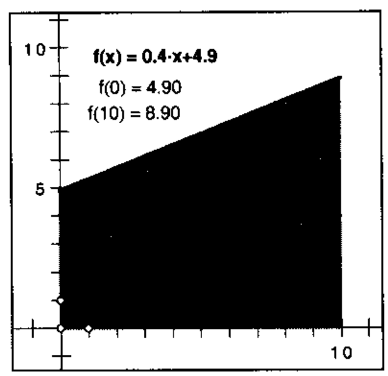


Sketch and Investigate

1. **Open** the document **AreaLimits.gsp** in the **Exploring Limits** folder. On page 1 there is a constant function $f(x) = 4.9$ on the domain $x = 0$ to $x = 10$.

Q1 Using geometry, find the area of the shaded region under the curve $f(x) = 4.9$ from $x = 0$ to $x = 10$.

2. Go to page 2 of the document. On this page, $f(x) = 4.9$ at $x = 0$, but this time, f increases at a constant rate on its domain $[0, 10]$ to a value of 8.9 at $x = 10$.



Q2 Calculate the area of this shaded region by using the area formula for a trapezoid.

Unfortunately it's not always possible to find the number of square units of area by using a formula from geometry.

3. Press the *Next Page* button to go to page 3. On this page, you will find the plot of the function $f(x) = 0.1(x - 3)^2 + 4$.

Q3 This function has the same y-values as the previous linear function at $x = 0$ and $x = 10$. How do you think the area of the shaded region will compare to the area of the region on the previous page?

If your shaded region moves outside of the window, drag an axis or the origin until you can see the whole region.

4. Drag the unit point at (1, 0) to the right until there is a grid line at every integer. Use the grid to count the "boxes" (squares of area 1) and estimate the area of the region.

Q4 What is your estimate for the area of the shaded region if you use only whole boxes? Is this an overestimate or an underestimate?

Q5 What could you do to make a better or more accurate estimate?

5. Press the *Next Page* button to see a really large overestimate—one rectangle with a base of 10 units and a height of 8.9 units.

Here we have one rectangle on the interval [0, 10] and n defined as the number of rectangles, so here $n = 1$. The right endpoint of the interval [0, 10] is used to find the height, which is the y-coordinate at $x = 10$, or $f(10)$, so this rectangle is called a *right rectangle*. If we use the left endpoint of the interval ($x = 0$) to find the height, or $f(0)$, the rectangle is called a *left rectangle*.

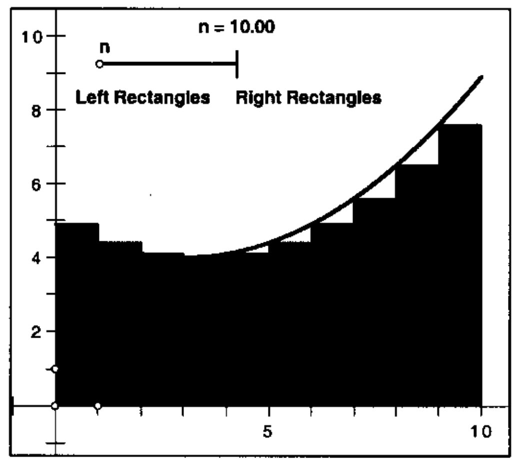
Point n on the left of the bar will make left rectangles.

6. Drag the slider for n to the left, past the bar, until $n = 1$ to show one rectangle using the left endpoint of the interval. This gives an estimate that is much too small—a rectangle with a base of 10 units, but this time with a height of 4.9 units.

7. To make rectangles of width 1, adjust your slider for left rectangles to $n = 10$.

8. Press the *Show Sample Point* button. Point C moves along the tops of the rectangles. Try moving point C left and right.

9. Measure the y-coordinate of point C by selecting point C and then choosing **Ordinate (y)** from the Measure menu.



Your estimate should be the same as the value for *Area Sum*.

Q6 Use the y-coordinate of point C to calculate the area of each of the rectangles. Write down their areas and find the sum. What is your estimate for this case? Is it an underestimate or an overestimate?

10. Adjust your slider to show right rectangles with $n = 10$.

Save this answer to use in the Explore More Section.

Q7 Using right rectangles, what is the value for *Area Sum* in this case? Given just the expression for the function f, how could you find this sum? Is it an underestimate or an overestimate?

Using rectangles to estimate area, as you did above, will give you one overestimate and one underestimate. They can't both be overestimates or underestimates.

Q8 Back in steps 5–6, you looked at rectangles with a base of 10 units. One gave an underestimate of 49 square units, and the other gave an overestimate of 89 square units—a difference of 40 units. How far apart are your two estimates in Q6 and Q7 using rectangles with a base of 1 unit?

Q9 What do you think will happen to the difference between the estimates if you use rectangles of width 0.5?

11. Adjust the slider to show right rectangles with $n = 20$ and record the value of *Area Sum*. Then look at left rectangles with $n = 20$.

Q10 What is the difference between the estimates? Is it what you expected?

Did you find that increasing the number of subdivisions by a factor of 10 caused a reduction in the difference between the estimates by a factor of 10? Did doubling the number of subdivisions to 20 result in a further halving of the difference?

Q11 Use the slider to find the difference between the left and right rectangle sums for $n = 25$ or some other value. Explain how the difference between the estimates is related to the value of n. Can you explain why this relationship holds?

To see why this happens, press the *Next Page* button to go to page 5.

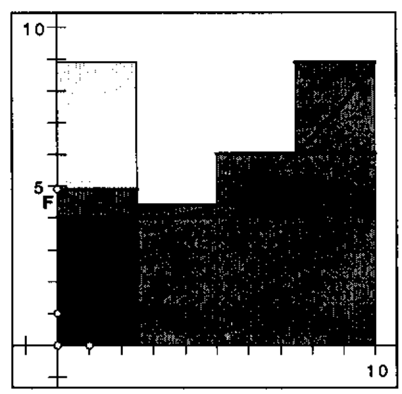
The sum of the areas of the rectangles shown when we're using the left endpoints to find the heights of the rectangles is called the *left sum*. When we use the right endpoints, it is called the *right sum*.

12. Experiment with the buttons to see just the left sum, then just the right sum, and then both sums at once.

13. While both sums are showing, adjust the slider so that $n = 1$. Then press the *Show Difference* button.

14. Check to see that when $n = 1$, you see a rectangle with an area of 40 square units—the difference between the areas of the two rectangles.

15. Adjust the slider so that $n = 4$. Note that the left and right sums have area in common. Press the *MovePointF* button to place the difference rectangle on top of the first left sum rectangle.



Q12 Increase the value of n. What do you notice about the height of the difference rectangle? Can you explain why this happens? What happens to the width of the rectangle?

Q13 Explain how you could use limits to show that you could make the difference between the left and right sum estimates as small as you wish.

Explore More

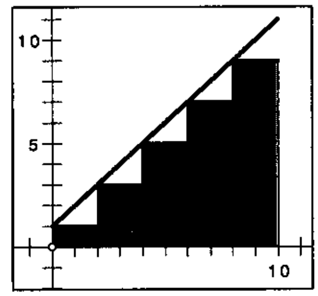
You can use limits to prove that the difference between the estimates approaches 0, but unfortunately, this does not tell you what area value the left and right sums approach. You will need a different limit for that.

Area and Limits (continued)

1. Go to page 6. On this page is the linear function $f(x) = x + 1$.

Q1 What is the exact area of the shaded region under the curve $f(x) = x + 1$ on the domain [0, 10]?

Q2 Write an expression to calculate the area of the left sum with 5 rectangles.



Q3 How would this expression change if there were 6, 7, 8, 10, 20, or even more rectangles? Write a general expression for the sum for n rectangles using Σ notation.

Q4 Using this expression, write a limit to express the area of the shaded region.

Q5 Write an expression for the sum of the 10 rectangles you made for the function $f(x) = 0.1(x - 3)^2 + 4$ back in Q7. How can you find the sum of these rectangles without adding everything up by hand?