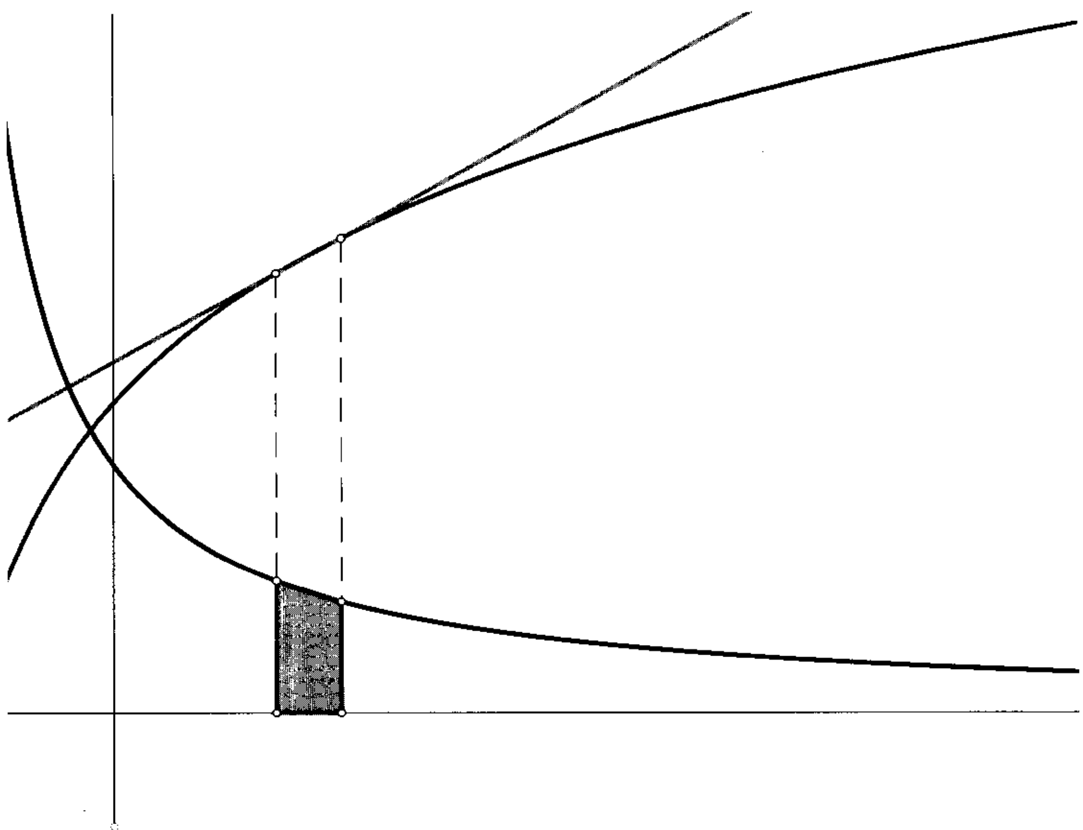
# 1

# Exploring Change



## Visualizing Change: Position

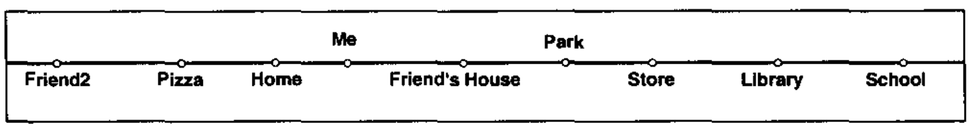
Name(s): ____________________

Calculus is the study of change and how to model that change—change in amounts, change in position, change in speeds . . . . A familiar situation to start with is distance from a starting point or home base. Suppose you woke up one morning and wanted to describe your travels for the day in terms of distance from your home. How would you do that? Depending on your route, that could get complicated fast. So let's start out easy and assume that every place you want to go, including your home, is conveniently located on a straight line. Now how would you do this?

### Sketch and Investigate

One way to begin a good model is to sketch your movement along a line.

1. **Open** the sketch **Position.gsp** in the **Exploring Change** folder. You will see a horizontal line with variable point *Me* which can move between home, school, and other important locations.




To talk about your *position* along a line in mathematics is to give the distance and direction from the starting point to point *Me*. To calculate your distance along a horizontal line, you subtract the $x$-coordinates of your points.

To measure the $x$-coordinate of a point, select the point with the **Arrow** tool and choose **Abscissa(x)** from the Measure menu.

2. Measure the $x$-coordinate of point *Me*. To calculate the difference $x_{Me} - x_{Home}$ choose **Calculate** from the Measure menu, and then click on each measurement to enter it into the calculator. Label this distance *Position* by choosing the **Text** tool and double-clicking on your measurement, $x_{Me} - x_{Home}$.

To turn on tracing, select point *Me*, then go to the Display menu and choose **Trace Point.**

3. Turn on tracing for point *Me* and drag along your line.

**Q1** Can you see your path?

**Q2** What happens to your *Position* measurement when you are at a location left of point *Home*? Right of point *Home*? When you are at point *Home*?

The measurement $x_{Me} - x_{Home}$ is called a *directed distance* because it gives both the direction and distance between the two points.

Your trace gives a reasonable model of your path, but it is limited to one dimension. If you look at this trace at the end of the day, can you tell what you did? If you traveled back and forth at all, some information was lost. A better model is a *position plot*, which uses time to expand this line into two dimensions by plotting position versus time. Time is simulated here by the $x$-coordinate of a point moving on an axis.

4. Press the *Show Time* button to show the time axis. Select point *Time* and measure its *x*-coordinate. Label this measurement *Time*.

5. Select the measurements *Time* and *Position*, in that order, and choose **Plot as (x, y)** from the Graph menu. Label this point *P*. So the *x*-coordinate of your point *P* will be time, and the *y*-coordinate will be your position.

To change the color or width of an object, select the object, then go to **Color** or **Line Width** in the Display menu.

6. Use the **Arrow** tool to select point *P*. Color it with a bright color and turn on tracing for point *P*.

Press the *Wake Up* button to put points *Me, Time,* and *P* at their starting points. For this first trial, you won't move point *Me* at all.

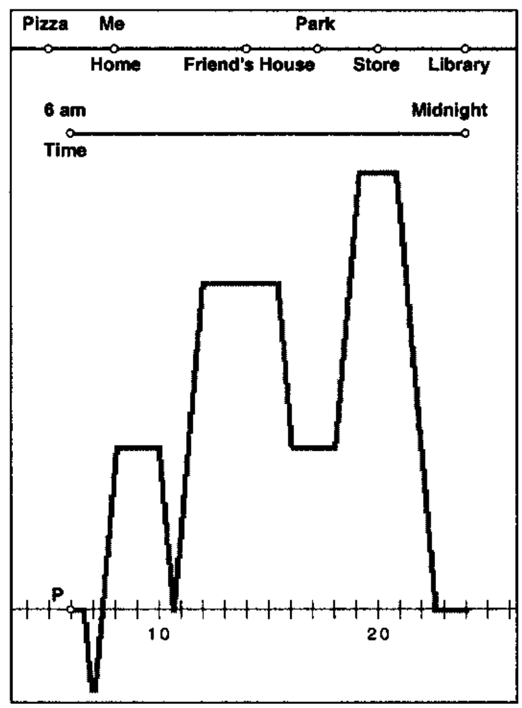
**Q3** What kind of position trace do you think you'll get if point *Me* just sits there and point *Time* moves? (Think! How is your distance from home changing if you stay in your bed as the minutes tick by?)

You can control the speed of animation. Use the Display menu to show the Motion Controller, and then change the speed.

7. Check your answer by pressing the *Animate* button. Observe for awhile, and then press it again to stop.

8. After examining your trace (was it what you expected?) press the *Wake Up* button and choose **Erase Traces** from the Display menu.

Now drag point *Me* while point *Time* moves. Point *P* will plot your position with respect to time. You can drag point *Me* any way you wish, but try each of these suggested movements as well. Each time, draw a little sketch of your trace in the margin and make a note of what you did to get that trace. Remember to press *Wake Up* and choose **Erase Traces** when you want to try a new motion.

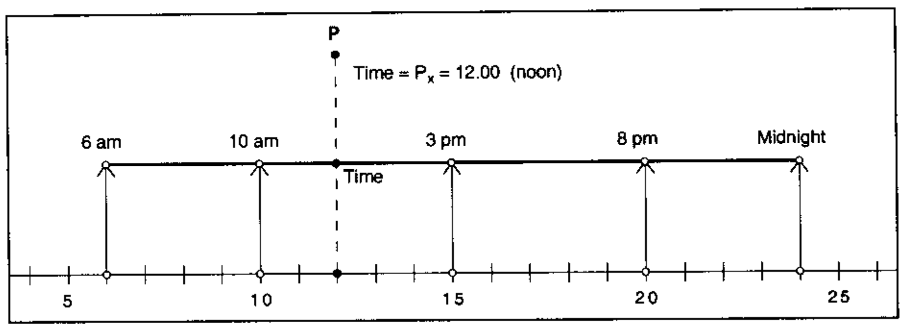



A. Drag point *Me* at a steady pace to your friend's house (try not to vary your speed at all), stay for about an hour, and then go home at a steady pace.

B. Stay home until 8 AM, and then start out at a slow, steady pace to the right. Speed up gradually until you get to school. Turn around and head toward home at the same quick pace. As you get closer to home, gradually slow down to a crawl.

C. Stay home until 7 AM, and then start out quickly toward school, slowing down more and more as you get closer. Once you get to school, tell the nurse you're sick and start home slowly, speeding up as you get closer to home.

D. Leave home at 7 AM and get to your friend's house about 7:30. Leave for school from your friend's house in time to stop briefly at the store for a snack and still get to school by 8:15. Leave school at 3 PM (15:00), hang out at the store for a half hour, then go to your friend's house for dinner, then off to the library to study for an hour. Make sure you're home by 9 PM (21:00)—it's a school night!




E. Leave home at 9 AM (it's a holiday), pick up Friend1 at your friend's house and go to the store to get picnic supplies. Pick up Friend2 by 11 AM, go get a pizza and head to the park for lunch. Hang out until 2 PM (14:00), then head to the library where they're showing free movies at 3 PM (15:00). Head back home for dinner at 6 PM (18:00).

Make a variety of traces, and look for relationships between how you move the point and the resulting trace.

**Q4** How must you drag point *Me* to make the directed distance (position) measurement increase? What happens on the trace when you do this? (Remember that going from –2 to –1 is an increase.)

**Q5** How must you drag point *Me* to make the directed distance (position) measurement decrease? What happens on the trace when you do this?

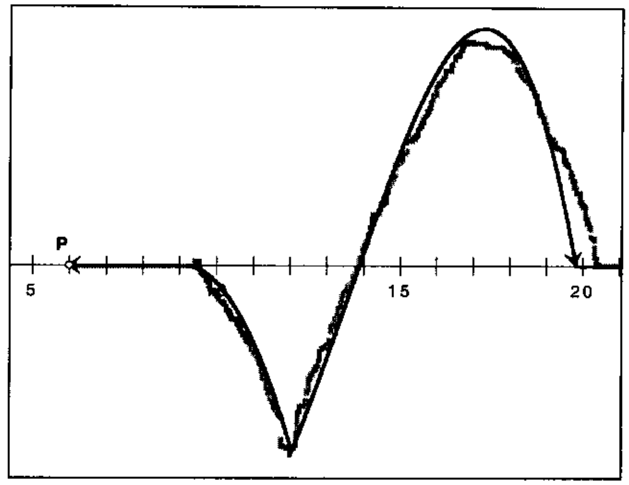
Press the *Saturday* button to show the position graph for a weekend day. Make a trace, trying to match the day's activities as closely as you can. During which part of your travels did you have to go the fastest? When did you move most slowly? When did you stay in one place?

Hide Saturday and press the *Sunday* button. Make a trace, trying to match the day's activities as closely as you can. During which part of your travels did you have to go the fastest? When did you move most slowly? When did you stay in one place? Where did you spend the night?

## Explore More

To go to a different page in a sketch, click on the page tab at the bottom left of the window.

→ Go to page 2 of the sketch. Here you will use the velocity control, rather than dragging point *Me* directly, to control the motion. To move point *Me* and point *P*, press the *Go* button and the *Animate* button. To stop point *Me*, press the *STOP!* button. To stop time, press the *Animate* button again. Experiment with the motion and compare the results of controlling velocity instead of controlling position directly. Try to trace both the Saturday path and the Sunday path. Which is easier to do using the *velocity* slider?

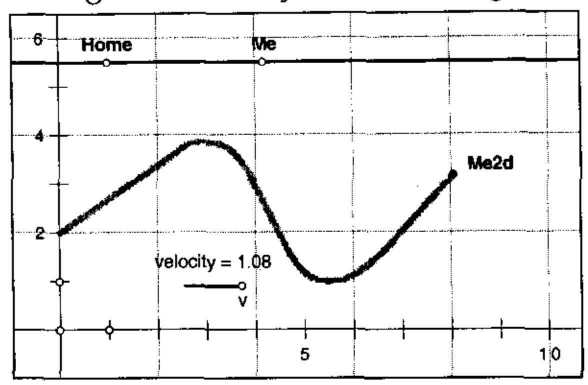


## Visualizing Change: Velocity

Name(s): ______________________

There are many ways to create motion or move an object. You could control where the object is located—its position—by dragging it around, or you could control how fast or slow the object moves—its speed.

Velocity is related to speed but it provides more information. If you know your velocity, you really know two things—how fast you are moving (speed) and the direction you are heading. Can knowing the velocity of an object tell you anything else? Are there any relationships or patterns between position and velocity? In this activity you will start to answer these questions by moving a point, controlling its velocity with a slider.




### Sketch and Investigate

1. **Open** the sketch **Velocity.gsp** in the **Exploring Change** folder.

You will see a horizontal line and point *Me* that moves along it. Point *Home* represents your base point or origin. You will also see the point *Me2d*. This point represents where you are at any time. The *x*-coordinate of point *Me2d* (labeled $time_{Me2d}$) represents time, and the *y*-coordinate (labeled $position_{Me2d}$) represents your position or distance from point *Me* to point *Home*.

2. Drag point *Me2d* around the plane, getting used to the way point *Me*'s position along the line (in other words, distance from point *Home*) relates to *Me2d*'s location in the time/position plane (in other words, its coordinates).

**Q1** Drag point *Me2d* horizontally. What happens to point *Me*? Explain.

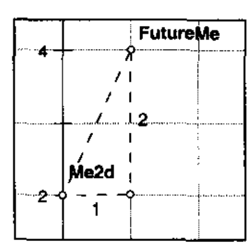
**Q2** You can drag point *Me2d* any way you'd like, but dragging in certain directions doesn't make sense given the way time works in our universe. How do you have to drag point *Me2d* so that it represents a physically possible motion of point *Me*?

Now we want to bring in velocity and see what effect it has.

3. Press the *Show Controls* button.

The grid in the figure is for clarification. It will not appear in the sketch.

You should see two sliders, one for velocity and one for a time interval. There is also a new point labeled *FutureMe*. This point is located one time interval away at the position you would reach *if* your velocity stayed constant. The *deltaT* slider is set at 1 and the *velocity*



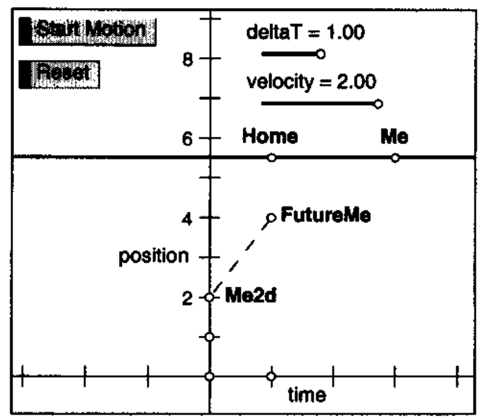
slider should be set at 2. So point *FutureMe* should be to the right 1 unit and up 2 units.

**Q3** If you change *deltaT* to 0.5 and keep the velocity the same, what will happen to point *FutureMe*? Try it and see.

**Q4** Move the *deltaT* slider to various time intervals. Does point *FutureMe* move in any particular pattern? What happens to point *Me* or point *Me2d* when you change just the time interval? Why is that?

**Q5** Set *deltaT* back to 1 and now move the *velocity* slider to various values. Does point *FutureMe* move in any particular pattern? What happens to point *Me* or point *Me2d* when you change just the velocity? Why?

The *Start Motion* button will start both points moving in relationship to the set *velocity* and *time* intervals.




Select point *Me2d*, then choose **Trace Point** from the Display menu. You can also change the color of your selected point and trace in the Color submenu of the Display menu.

4. Press the *Reset* button to move point *Me2d* to *time* = 0.
5. Turn on tracing for point *Me2d*.
6. Set the *velocity* slider to 2 and the *deltaT* slider back to 1.

For these first trials, you won't change the *velocity* slider once your point is moving. Predict what kind of position trace you'll get if your velocity (speed and direction) stays the same. Sketch this prediction in the margin.

**Q6** Press the *Start Motion* button and observe point *Me*'s motion and point *Me2d*'s corresponding time/position trace. Press the button again to stop the motion. Describe your trace. (Was it what you predicted?)

**Q7** Press the *Reset* button, but do not clear your trace. Instead, change the *velocity* slider to 0.5 and make point *Me2d* a different color. Make a prediction, and then press the *Start Motion* button again. What happened this time? How are your traces different? How are they the same?

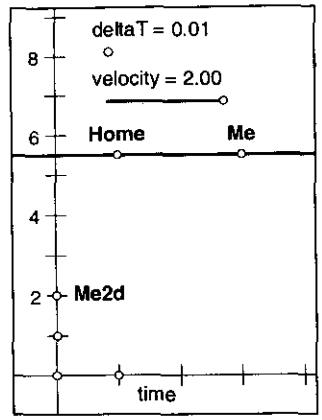
**Q8** Repeat Q7, but this time set your *velocity* slider to a negative value. Any idea what will happen? Press the *Start Motion* button again. What happened this time? How are your traces different? How are they the same?

**Q9** What conclusions can you reach about movement and position traces when velocity is constant over a time interval?

**Q10** What are the equations for the different traces you see on your screen? What would the equation for the trace be if velocity were set to 0?

For the next set of trials, you will change the velocity of point *Me* while time is changing. The smaller the time interval, the more accurate the trace, so set *deltaT* as close to 0.1 as possible and hide point *FutureMe*. You can change the *velocity* slider to any value you wish, but try each of these suggested experiments as well. For each experiment, draw a little sketch of your trace in the margin. Remember to choose **Erase Traces** from the Display menu and press the *Reset* button when you want to start over.

To hide a point, select the point and then choose **Hide Plotted Point** from the Display menu.




A. Start with the velocity at a positive value. Increase the velocity, and then decrease the velocity, but keep it positive throughout the experiment.

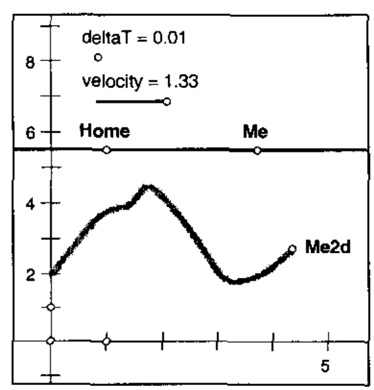
B. Start with the velocity at a negative value. Increase the velocity, and then decrease the velocity, but keep it negative throughout the experiment. (Remember that –2 to –1 is an increase!)

C. Start with *velocity* > 2. Decrease the velocity, and then increase it. Again, keep the velocity positive throughout.

D. Start with –1 < *velocity* < 0. Decrease the velocity, and then increase it, but again, keep the velocity negative throughout.

E. Start with a positive velocity and decrease to a negative value. Then increase the velocity again until you get to 0. Stay at 0 for a while and then increase the velocity again.

**Q11** How are the traces in A and B similar? How are they different? What happens to the position trace when you switch from increasing the velocity to decreasing it?




**Q12** How are the traces in C and D similar? How are they different? What happens to the position trace when you switch from decreasing the velocity to increasing it?

**Q13** How are the traces in A and C similar? How are they different? What about B and D?

**Q14** What happened when you changed the velocity from positive to negative? From negative to positive? What happened when you stayed at *velocity* = 0?

**Q15** For each of the following, describe the position trace that you would get. Then check your answer using the *velocity* slider.

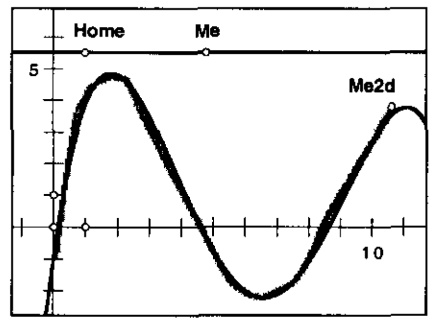
a. positive and increasing velocity

b. negative and increasing velocity

c. positive and decreasing velocity

d. negative and decreasing velocity

## Explore More




Go to page 2 of the sketch. Press the *Show Path1* button. Using your answers from Q15 for reference, make a trace trying to match the path as closely as you can. During which part of your trace did you have to go the fastest? When did you move the slowest?

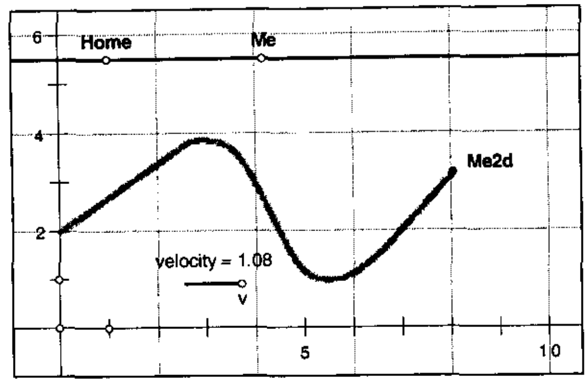
Hide Path 1 and press the *Show Path2* button. Again, try to match the path as closely as you can.

What is different about Path 2? Which one was easier to trace? Is it possible to trace Path 2's corners?

## Can You Predict the Trace?

Name(s): ____________________

In the previous activity you created a position plot by controlling the velocity of the object. You saw that what you did with the velocity determined what the position trace looked like. Could you look at a position plot and predict the velocity plot? In this activity, you will trace position and velocity at the same time and use what you discover to predict what one graph would look like given the other.

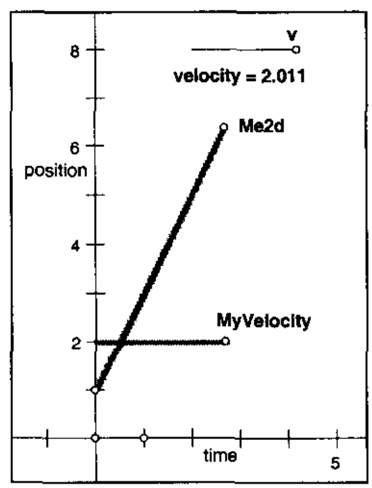



### Sketch and Investigate

1. **Open** the sketch **RemoteControl.gsp** in the **Exploring Change** folder. (To see how this sketch works, review pages 7–9.)
2. Select the measurements of time and velocity, in that order, and choose **Plot as (x, y)** from the Graph menu.
3. Turn on tracing for this new point and label it *MyVelocity*.
4. Press the *Move* button, but don't change the *velocity* slider. Just watch.

To stop the experiment, press the *Move* button again.

You should see two traces created—one for position and one for velocity. Since the velocity is not changing, its trace is a horizontal line.




**Q1** The position trace is linear; what is its slope? Have you seen this value anywhere else?

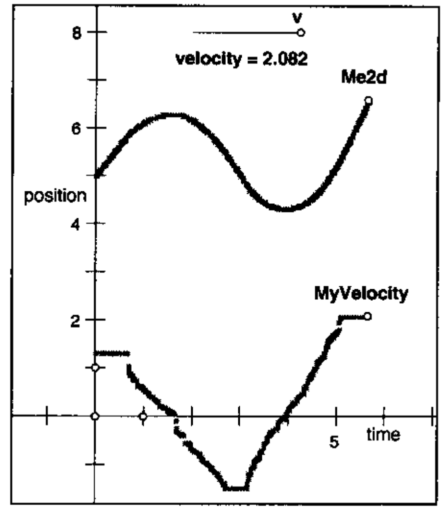
5. Press the *Reset* button and choose **Erase Traces** from the Display menu to return point *MyVelocity* and point *Me2d* to their original locations so you can run a new experiment.

This time, change the velocity after you press the *Move* button. Increase or decrease the velocity as you wish, but try each of these suggested experiments as well. For each experiment, draw a little sketch of your trace in the margin. Remember to press the *Reset* button and choose **Erase Traces** from the Display menu when you want to start over.

A. Start with the velocity at a positive value. Increase the velocity, and then decrease the velocity, but keep it positive throughout the experiment.

B. Start with the velocity at a negative value. Increase the velocity, and then decrease the velocity, but keep it negative throughout the experiment. (Remember that –2 to –1 is an increase!)

C. Start with *velocity* > 2. Decrease the velocity, and then increase it—again keep the velocity positive throughout.




D. Start with –1 < *velocity* < 0. Decrease the velocity, and then increase it—but again keep the velocity negative throughout.

E. Start with a positive velocity and decrease to a negative value. Then increase the velocity again until you get to 0. Stay at 0 for a while and then increase the velocity again.

**Q2** When your position trace is increasing (going up), what can you say about your velocity trace? (Or what must you do with the *velocity* slider to make the position trace keep going up?)

**Q3** When your position trace is decreasing (going down), what can you say about your velocity trace? (Or what must you do with the *velocity* slider to make the position trace keep going down?)

**Q4** When your position trace hits a maximum or minimum, what happens to the velocity trace? (Or what must you do to make your position trace change direction?)

**Q5** When your position trace is *concave up* (curved up), what can you say about your velocity trace? (Or what must you do with the *velocity* slider to make the position trace curve upward?)

**Q6** When your position trace is *concave down* (curved down), what can you say about your velocity trace? (Or what must you do with the *velocity* slider to make the position trace curve downward?)

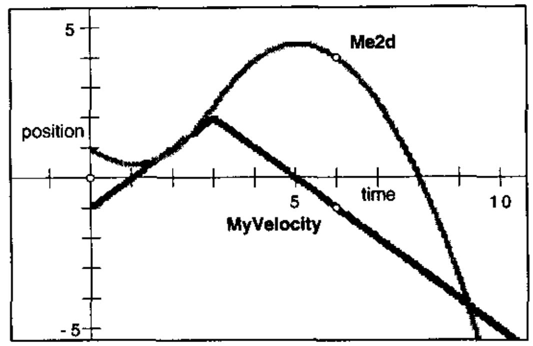
These questions suggest some connections between the plot of the position of a point and a plot of the velocity of the point. On the second page of the document, you can try predicting the trace of velocity from the trace of position.

6. Press the *Predict Velocity Trace* button to go to page 2.

On this page, the velocity slider will be controlled by a hidden function. Instead of adjusting the velocity slider, you can watch the velocity slider and the position trace and make a prediction about what the velocity trace would look like.

7. Press the *Move* button and observe the position trace and the velocity slider. The plotted point that tracks the velocity is hidden.

8. Make a sketch in the margin of what you think the velocity trace will look like, using your answers to the questions above. After you make your prediction, press the *Reset* button, then the *Show Point MyVelocity* button, and then the *Move* button to see the trace of position and velocity at the same time.




**Q7** How does the velocity trace compare to your prediction in step 8?

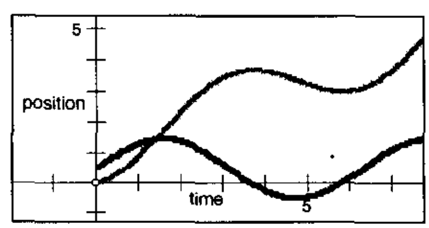
9. To see another example, press the *Next* button and repeat steps 7–8. You can practice on any of the eight examples on pages 2–9 in the document. When you can accurately predict the velocity trace from the given position trace for one example, go on to the next by pressing the *Next* button.

Next, you will try to predict the position trace given the velocity trace. First, consider the questions below. (You can use your answers to Q2–Q6 and page 1 of your document to answer these questions.)

**Q8** When your velocity is positive, what can you say about your position trace?

**Q9** When your velocity is negative, what can you say about your position trace?

**Q10** When your velocity trace hits a maximum or minimum, what happens to the position trace? (What happens to the position trace when the velocity slider changes direction?)




**Q11** When your velocity trace is increasing, what can you say about your position trace? (What happens to the position trace when the velocity slider moves to the right?)

**Q12** When your velocity trace is decreasing, what can you say about your position trace? (What happens to the position trace when the velocity slider moves to the left?)

These questions suggest some connections between the plot of the velocity of a point and a plot of the position of the point.

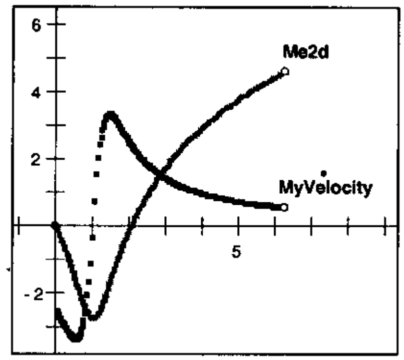
The *Predict Position Trace* button is on page 1 of your document.

10. Press the *Predict Position Trace* button to go to page 10.

Point *Me2d* is hidden on this page. Here you'll use the velocity trace and the position of point *Me* on the line in relation to point *Home* to make a prediction about what the position trace would look like.

11. Press the *Move* button and observe the velocity trace and the position of point *Me*.

12. Make a sketch in the margin of what you think the position trace will look like, using your answers to the questions above. After you make your sketch, press the *Reset* button, then the *Show Point Me2d* button, and then the *Move* button to see the trace of position and velocity at the same time.




**Q13** How does the position trace compare to your prediction in step 12?

13. To see another example, press the *Next* button and repeat steps 11–12. You can practice on any of the six examples on pages 11–16 in the document. When you can accurately predict the position trace from the given velocity trace for one example, go on to the next by pressing the *Next* button.

### Explore More

Point *R* sets the starting location for point *Me2d* in the predicting position trace pages. Try moving this point, pressing *Reset*, and running the experiment again. (The velocity function will remain unchanged.) Compare the position traces for two different initial positions. What do you notice? Why do you think this happens?

## Catching the Point

Name(s): ____________________

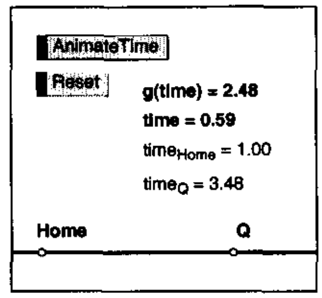
In the last two activities, you looked at position plots or motion by moving a point—either by dragging the point to various places or by controlling its velocity with a slider. You saw that there was some kind of relationship between the point's position and its velocity. What if the point moves according to some pattern that is not in your control? Is there still a relationship, and what does all this have to do with calculus? In this activity, you'll start to answer both questions.

### Sketch and Investigate

1. **Open** the sketch **Motion.gsp** in the **Exploring Change** folder. Point $Q$ moves on the horizontal line according to a pattern—a function that gives its directed distance from point *Home* at a specific time.

To measure the coordinates, choose **Abscissa(x)** from the Measure menu and then choose **Ordinate(y).** Then to calculate the distance, choose **Calculate** from the Measure menu, and click on each measurement to enter it into the calculator.

2. Measure the $x$-coordinates of point *Home* and point $Q$ and calculate the distance from point *Home* to point $Q$, or $time_Q - time_{Home}$. Label this measurement *position.*




Now press the *Animate Time* button—watch how the point moves and how fast it moves. Do this a few times—check that the *position* measurement and the function value match at *all* times, watch how the velocity changes, and try to figure out when point $Q$ changes direction. You can stop the animation at various times by pressing the *Animate Time* button again. This is most helpful for figuring out when the point changes direction.

**Q1** Describe point $Q$'s path. For what values of time does it go to the right? Go to the left? Stop?

**Q2** When does it seem to move the fastest? The slowest?

It's difficult to figure out these kind of questions when your movement is limited to one dimension—especially if you can't stop time. So creating a position plot of *position* versus *time* is the next step.

3. Select measurements *time* and *position,* in that order, and choose **Plot as (x, y)** from the Graph menu. Label this point $P$.

To turn on tracing, select point $P$, then go to the Display menu and choose **Trace Plotted Point.** You can also change the color of your selected point and trace by choosing **Color** in the Display menu.

4. Turn on tracing for point $P$ and color it a bright color.
5. Press the *Animate Time* button. After point $P$ traces one cycle, press it again to stop. Does point $P$ change direction at the same times as point $Q$? Do the two points seem to go at the same speed?

**Q3** Describe point $P$'s path—when does point $P$ go up? Go down? Change direction? Is there a relationship between your answers for point $Q$ and point $P$?

## Catching the Point (continued)

**Q4** Describe point *P*'s motion. When does point *P* move quickly—either up or down? When does it slow down?

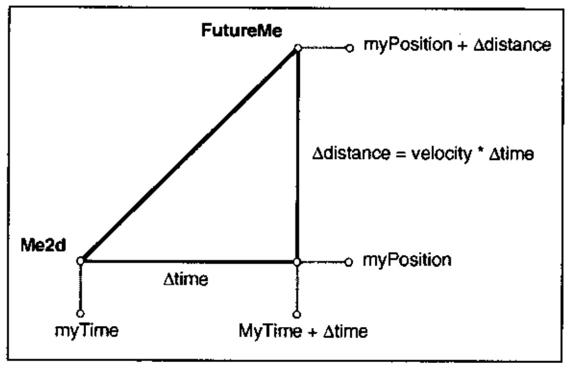
To figure out if there is a pattern or relationship between the velocity and the position of point *Q*, you are going to make a model by using an independent point, point *Me2d*, and some sliders. Your goal is to construct a point, *FutureMe*, that will help you control point *Me2d*'s path and velocity. Then you'll try to match point *P*'s path.

To construct a point, choose the **Point** tool, and then click anywhere in the sketch. To label the point, double-click on it with the **Text** tool.

6. Construct a point anywhere in your sketch using the **Point** tool. Label this point *Me2d* and measure its $x$- and $y$-coordinates. Label these measurements *myTime* and *myPosition*.

7. Press the *Show Sliders* button to show a time interval slider, labeled *deltaT*, and a slider for velocity.

Right now, you're sitting at the point (*myTime*, *myPosition*). If some time passes and you are moving with a constant velocity, where will point *FutureMe* be located? Use the diagram to help.




Choose **Calculate** from the Measure menu. Then click on each measurement to enter it into the calculator.

8. Calculate point *FutureMe*'s new $x$-coordinate. Then calculate its $y$-coordinate. Plot this point. (Hide its label.)

To construct a line segment, select each point then choose **Segment** from the Construct menu.

9. Construct the line segment between point *Me2d* and point *FutureMe*. This line segment represents your path, so color it a different color than point *P*.

**Q5** If you shorten or lengthen your time interval, *deltaT*, how does that affect your path?

**Q6** If you shorten or lengthen your velocity slider, how does that affect your path?

Be sure to deselect all objects first.

10. To start moving, select just point *Me2d* and then select point *FutureMe*. Choose **Edit | Action Buttons | Movement.** Select **slow** for the speed and label the button *Move Me.*

By pressing the *Move Me* button, you can experiment with using different time intervals and velocities to move your path.

Press the Esc key once to stop. Then, to start over, drag point *Me2d* back to a starting point and choose **Display | Erase Traces.** Then press the *Move Me* button.

11. Set $deltaT \approx 2$ and $velocity \approx 0.5$. Drag point *Me2d* to (0, 2) to start. Turn on tracing for your segment. Press the *Move Me* button. Keep your *deltaT* constant but experiment with your *velocity* slider.

**Q7** What happens to your path trace if you make a large change in your velocity?

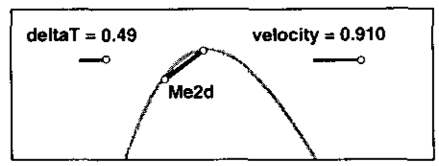
12. Set *deltaT* ≈ 0.1 and *velocity* ≈ 1. Drag point *Me2d* to (0, 2) to start. Erase all traces. Press the *Move Me* button. Again keep your *deltaT* constant but experiment with your *velocity* slider.

**Q8** Now what happens to your path trace if you make a large change in your velocity?

Because the point's construction assumed a constant velocity over a time interval, you'll want to make *deltaT* small so that you can change your velocity frequently. Your goal in the next part will be to use your *velocity* slider to trace a path as close to point *P*'s path as possible.

13. Press the *Show P's Path* button and set *deltaT* ≈ 0.50.

14. For now, turn off tracing for the segment by selecting the segment, and then deselecting **Trace Segment** on the Display menu. Erase all traces.




15. Drag point *Me2d* to anywhere on point *P*'s path and adjust your velocity until point *FutureMe* is also on the path. (This is roughly point *Q*'s velocity—note that it doesn't match the path exactly.)

16. Press the *Move Me* button and try to stay on point *P*'s path by using your *velocity* slider.

17. Try several more times, each time decreasing your time interval by 0.10, but don't go below 0.

**Q9** Does your accuracy improve as *deltaT* gets closer to 0? Can you stay on point *P*'s path the whole time?

**Q10** What must you do with your *velocity* slider in order to make your path go up? Go down? Change direction?

Once you can stay on point *P*'s path most of the time, you're ready to model the velocity. You'll do this by making a new trace.

*deltaT* should still be close to zero.

18. Press the *Reset* button, then drag your point *Me2d* back to point *P*'s starting point, (0, –1). Line up point *FutureMe* with point *P*'s path by increasing or decreasing your *velocity* slider.

19. Once you are lined up on point *P*'s path, select *myTime* and *velocity* in that order and choose **Plot as (x, y)** from the Graph menu. This will track the velocity of point *Me2d*.

You may need to scroll up your window in order to find your velocity point. After you label it and turn on tracing, scroll back down.

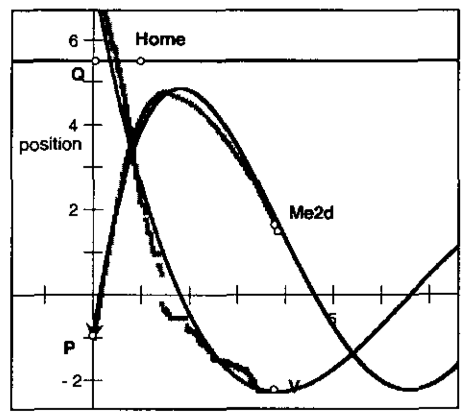
20. Label this point *V*, turn on tracing, and color point *V* a new bright color.

21. Press the *Move Me* button and make sure you stay on point *P*'s path as you did before. (You can start over if you "fall off.")

22. Stop and examine your new trace. Do the patterns between the position and velocity functions hold from the previous activity? (See the Activity Notes if you didn't do the previous activity.)

So how does calculus come in? The little segment path you traced really used the average velocity in its construction because we assumed the velocity stayed constant over a time interval. In calculus we ask, "What is the velocity *now*—in this instant or at this moment?" This is called *instantaneous velocity.* You might think we can't use the average velocity at all to answer this question because the average velocity is defined as the change in distance divided by the change in time, and if we're talking about an instant, the change in time is 0. That would give us a 0 in the denominator—a major problem! One way to start thinking about this is to ask the question, "What happens if I let *deltaT* get really close to 0 or *go to* 0, but never really get there—what will I see?" Calculus answers this question using a concept called limits.

Press the *Show Q's Velocity* button. This is point *Q*'s instantaneous velocity as a function of time. How close did your velocity trace come to matching this? Beside the jerkiness of a trace, if you did a fairly good job of staying on point *P*'s path and had a very tiny *deltaT*, your velocity trace should be fairly close to the plotted velocity.




## Explore More

Press the *Hide Q's Velocity* button. Select point *P*'s path. Choose **Properties** from the Edit menu. Go to the Object panel and choose Function g from the Parents menu. Deselect the Hidden box and click OK. The function highlighted in the sketch is the one that moves point *P* and point *Q*. Double-click on this function and change it to $g(x) = \text{abs}(x^2 - 12x + 32)$.

Try to trace this function's path. How is this path different from the previous function's path? How does that affect the velocity?

Once you have a velocity trace that you are satisfied with, press the *Show Q's Velocity* button. Select the velocity plot, then choose **Properties** from the Edit menu. On the Plot panel, change **continuous** to **discrete.** Click OK. Compare your trace of the velocity to the plot. How did you do?

# Plotting Average Rates of Change

**Name(s):** ______________________

In this activity, you will examine two ways of describing average rates of change. In the process, you will be introduced to two new tools and investigate how to apply them to various functions as you create an average rate of change plot. (If you would like to build these tools yourself, go to **Tool Tips.doc** in the **Tools** folder.)

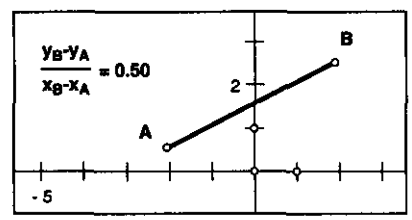
## Sketch and Investigate

1. **Open** the document **Step.gsp** in the **Exploring Change** folder.
2. Plot two points anywhere in the coordinate plane using the **Point** tool. Label them point $A$ and point $B$ with your **Text** tool.

For any two points $A$ $(x_A, y_A)$ and $B$ $(x_B, y_B)$, the average rate of change of $y$ with respect to $x$ is expressed by the ratio $\dfrac{y_B - y_A}{x_B - x_A}$.

The **Custom** tools icon is at the bottom of the toolbar.

3. Choose **Average Rate** from **Custom** tools. Click on point $A$ and then point $B$.




This tool calculates the average rate of change from point $A$ to point $B$. Drag points $A$ and $B$ around to observe the changes in the ratio.

**Q1** When is the average rate of change from point $A$ to point $B$ positive? When is it negative?

**Q2** Does point $A$ need to be on a particular side of point $B$ in order to have a positive rate of change? Why or why not?

Be sure not to select points $A$ and $B$ here.

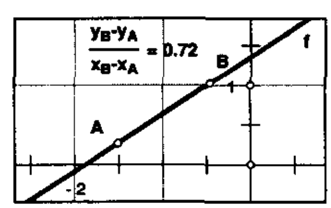
4. Select segment $AB$. Choose **Slope** from the Measure menu.

**Q3** Drag point $A$ and point $B$ around to observe the changes in both measurements. Why do they always match? (Move point $B$ back to the right of point $A$ when you're done.)

Now that you have found the rate of change between two points, you will look at a series of points on a function plot.

To enter $a$ and $b$ into the calculator, click on their measurements.

5. Using the measurements provided, plot the function $f(x) = ax + b$ by choosing **Plot New Function** from the Graph menu.




6. Click on an empty spot in the sketch to deselect $f(x)$.
7. Select point $A$ and then the plot of $f(x)$—not the measurement. Then choose **Merge Point to Function Plot** from the Edit menu. Merge point $B$ to the function plot as well.

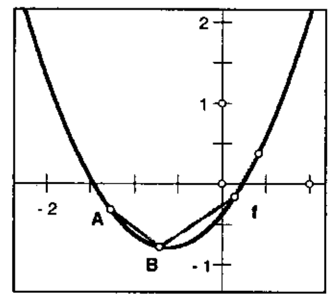
8. Choose **Average Rate** from **Custom** tools, click on point *B*, and then click on the function plot to the right of point *B* to make a new point.

9. Calculate the average rate of change on the next adjacent interval. (With the **Average Rate** tool, click on the point you made in step 8, and then on a new point on the function to the right of that point. Interval lengths can be different.)

**Q4** Why are all your average rate measurements equal to each other? Does moving any of the points you created change that?

If you can't see your function plot, drag the slider for *c* until it comes into view.

10. Double-click on the expression for the function $f(x)$. Change $ax + b$ to $ax^2 + bx + c$.




**Q5** Are all the average rate measurements still equal to each other?

**Q6** How can you determine from the function plot where the average rate of change between two points on that function will be positive, negative, or 0?

As you saw in Q3, the average rate of change between two points can be seen graphically by the slope of a segment between those points. Can slope be plotted as a function as well? Slope is defined over an interval, so the question is, how do you plot a quantity defined over an interval? One way is with a step function.

11. Go to page 2 of the document.

12. Choose **Plot Slope** from **Custom** tools, and then click on points *A* and *B* in that order. This tool plots the slope between two points *A* and *B* on the interval $[x_A, x_B]$ as a "step"—or horizontal segment.

**Q7** Experiment with this "step" by dragging either point *A* or point *B* around in the plane. When does the step have a negative $y$-value? When does it have a positive $y$-value? When is it 0?

Go back to page 1 to use the **Plot Slope** tool on the points you created.

When creating a new step, click first on the point you just created for the previous step. You will be clicking twice on each endpoint where two segments meet.

13. With the **Plot Slope** tool, click on *both* endpoints of each segment you created on $f(x)$ in steps 8–9. Then use your tool to create more points as you did in step 9, until you have a total of eight "steps."

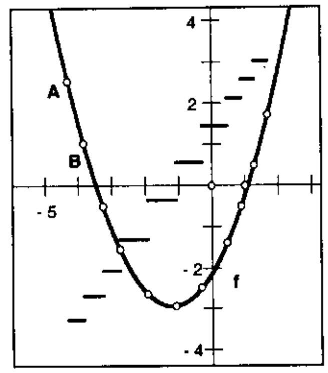
**Q8** Your "steps" make up a new relation that plots the average rate of change for each interval that you created on $f(x)$. With the **Arrow** tool, press the *Case a = 0* button to set $a = 0$. Explain why this new function plot of "steps" is a horizontal line.

**Q9** Experiment with the slider for *c*. Describe what happens to your step function as a result and explain why.

**Q10** Experiment with the slider for $b$. Describe what happens to your step function as a result and explain why.

Again, if you can't see your function plot, drag the slider for *c* until it comes into view.

14. Make your measurement for $a$ greater than 0.




**Q11** In the figure to the right, the steps go up from left to right. Why? How can you adjust your function so that they go down?

**Q12** Predict what will happen if you again adjust the value for $c$. Explain why this happens. Check your answer.

You now have a new function that plots the average rate of change for each interval that you created on the function $f(x)$.

**Q13** Go to page 3 and predict what the step function will look like for that cubic. How can you determine from the function plot when the rate of change step function will be positive (above the $x$-axis), negative (below the $x$-axis), or zero?

Press the *Show Steps* button. How did you do?

## Explore More

On page 4 of the document you will find the plot of a function, with two points $A$ and $B$ on the function. Here the slider labeled $h$ controls the distance between the $x$-coordinates of the two points.

1. Using the **Plot Slope** tool from **Custom** tools, click on points $A$ and $B$ to create a "step" of the rate of change on the interval from point $A$ to point $B$.
2. With the segment still selected, choose **Trace Segment** from the Display menu.
3. When you press the *MoveA→B* button, the two points will move to the next adjacent interval. Press the button a few more times to create a trace of the step function.

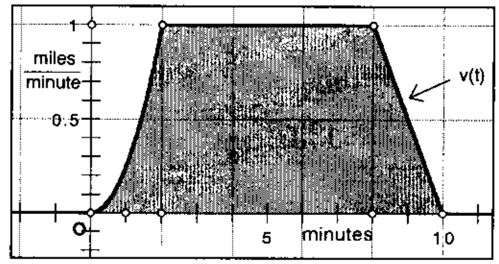
To edit the speed, double-click on the *MoveA→B* button with your **Text** tool and go to the Move panel.

4. Adjust your slider so that the measurement $h \approx 0.1$ and edit the *MoveA→B* button so that the points move at **medium** speed.
5. Drag point $A$ to the left side of the window and choose **Erase Traces** from the Display menu. Then press the *MoveA→B* button. Press it again to stop.

**Q1** Try this for a number of functions, and record your observations. Can you predict what the trace of the average rate of change "step" will look like, given the plot of the function?

## Going the Distance

**Name(s):** ____________________




When you drive a car, you can always find your velocity just by looking at your speedometer. Can you figure out your distance from that? Yes, but only if you are driving at a constant speed. But that is nearly impossible, and you still have to accelerate when you start and decelerate when you stop. At these times, your velocity isn't constant at all. The sketch above depicts a possible velocity function that fits these conditions. (Note that here we're assuming you can drive at a constant velocity—we'll pretend you have perfect cruise control.) Given this velocity function, how could you find your total distance? Why is the region below the function shaded in? If you've guessed that the two questions are probably related, you're right. Let's find out why.

### Sketch and Investigate

1. **Open** the sketch **Area1.gsp** in the **Exploring Change** folder. Here you have the velocity function shown above, without the shading.

**Q1** For how many minutes is your velocity constant? What is your velocity for those minutes?

**Q2** Because your velocity is constant for this segment of your trip, you can figure out your total distance for this interval of time. What is this distance?

Be sure to select points *A*, *B*, *C*, and *D* in that order.

2. Construct the interior of rectangle *ABCD* by selecting the vertices of the rectangle and choosing **Construct | Quadrilateral Interior.**

Sketchpad's Area command uses inches, which is not relevant here, so calculate this by hand.

**Q3** What is the area of rectangle *ABCD*? Include units in your answer!

**Q4** Are the answers to Q2 and Q3 related in any way? Can you jump to any conclusions at this point?

Look at just the units of the axes—the horizontal axis is in minutes and the vertical axis is in miles per minute. So the units for any area involving a base and a height will be

$$base \cdot height = minute \cdot \frac{miles}{minute} = miles$$

But miles is purely a distance measurement, so in this case, the area under the velocity curve from $x = 2$ to $x = 8$ is the total distance traveled in that time. Is that always true? Maybe. Let's look at another example and see whether this conjecture holds. Consider the velocity function from $x = 8$ to $x = 10$. Here, instead of being constant, the velocity is a linear function that goes from 1 mile/min at $x = 8$ to 0 mile/min at $x = 10$. Here, too, you can calculate your total distance using your average velocity. (Why?)

**Q5** What is the average velocity on the interval [8, 10]? Using this, what is your total distance from $x = 8$ to $x = 10$?

Color this triangle a bright new color from the Color submenu of the Display menu.

3. Construct the interior of triangle *BCE* by selecting the vertices of the triangle and choosing **Triangle Interior** from the Construct menu.

**Q6** What is the area of triangle *BCE*? Include units in your answer.

Again, the area under this velocity curve from $x = 8$ to $x = 10$ is the total distance traveled in that time. Is this true all the time? Actually, it is, but the proof of that is a bit off in the future. For now, what's important is that the area under a velocity curve over an interval gives the total distance traveled in that interval. So you can find your total distance if you can calculate an area. Sounds simple, right? Then how are you going to calculate the area under the velocity curve from $x = 0$ to $x = 2$?

**Q7** What familiar polygon most closely approximates this area?

**Q8** Find an approximation for the total distance by finding the area of the polygon you chose in Q7. (Save this answer.)

4. Construct the interior of your polygon.

**Q9** Comparing the interior you constructed in step 4 and the actual region under the curve, is your approximation in Q8 too big or too small?

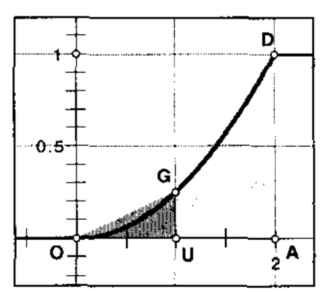
5. Select the interior of the polygon and choose **Display | Hide Triangle.**

To get a closer approximation, you could divide the interval [0, 2] into two sub-intervals and use two different polygons.

6. Select the unit point at $x = 1$ and measure its $x$-coordinate by choosing **Abscissa(x)** from the Measure menu.
7. Press the *Show Function* button and calculate $f(x_U)$ by choosing **Calculate** from the Measure menu. Click on the expression for $f$ and the measure for $x_U$ to enter them into the calculator.

Double-click on the point with the **Text** tool to change a label.

8. Select $x_U$ and $f(x_U)$, in that order, and choose **Plot as (x, y)** from the Graph menu. Label this point *G*.
9. Choose any of the light color tools from **Custom** tools and click on points *O*, *U*, and *G*, and then on point *O* again to construct the interior of triangle *OUG*.




10. Choose a different color tool from **Custom** tools and click on points *U*, *A*, *D*, and *G* to construct the interior of the trapezoid *UADG*.

**Q10** What are the areas of triangle *OUG* and trapezoid *UADG*? (Recall that the area of a trapezoid is $A = 0.5 \cdot base \cdot (height_1 + height_2)$.)

**Q11** What is your total area approximation? When you compare your shaded regions with the area under the curve, is this total area too big or too small? (Save this answer.)

**Q12** Did your approximation improve significantly? (Is a lot less of the shaded region outside of the curve now?)

Press the *Show Area Tools* button. On the *time* axis, there are three new points—*start*, *finish*, and *P*. Points *P* and *start* should be at the origin. Point *P* will sweep out the actual area under the curve from point *start* to point *finish*, so point *finish* is as close as possible to $x = 2$. The measurement *Area* should be 0 because point *P* has not moved yet.

11. Press the *Move P* button to calculate the actual area under the *velocity* curve and to shade in that region.

**Q13** What is the actual area under the curve from $x = 0$ to $x = 2$?

**Q14** Calculate your approximation errors from Q8 and Q11. Was your second approximation a significant improvement?

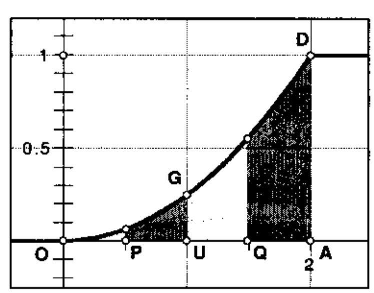
One way to tell whether your second approximation is a significant improvement is to calculate the percentage errors—divide your errors in Q14 by the actual area. You should have a drop from about 50% error to about 12% error. That is significant!

**Q15** Using your answers to Q3, Q6, and Q13, what is the total area under the velocity curve from $x = 0$ to $x = 10$? This is the total distance you traveled in those 10 minutes!

## Exploration 1

To get an even better approximation, you can do the same procedure again. Divide the region on [0, 2] into fourths, using one triangle and three trapezoids.

1. Using the **Point** tool, construct new points at $x = 0.5$ and $x = 1.5$ or as close to those values as possible. Label these points *P* and *Q*.




2. Construct the *y*-values on the function for these two points as you did in step 6 above. Then plot the points on the function at $x = 0.5$ and $x = 1.5$ as you did in step 8.

3. Construct the interior of the triangle on the interval [0, 0.5], using a color tool as you did in step 9. Calculate its area—this time with your measurements $x_P$ and $f(x_P)$ and the calculator.

4. Construct the interiors of the trapezoids over the intervals [0.5, 1], [1, 1.5], and [1.5, 2]. Use a different color tool for each one. Calculate the areas of the three trapezoids using your measurements.

5. Calculate a new approximation for the area under the curve from $x = 0$ to $x = 2$ by making a new measurement that sums the calculations you did in steps 3 and 4 above.

**Q1** What is your new approximation for the area under the curve from $x = 0$ to $x = 2$?

**Q2** What is the error for your new approximation? What is the percentage error? Was this new approximation a significant improvement?

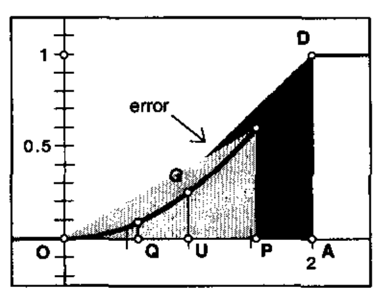
To get an even better approximation, you can do the same procedure again. Divide the region on [0, 2] into eighths, using one triangle and seven trapezoids. Then do it again by dividing your region into sixteenths and so on. This can quickly become tedious, so in the activity "The Trapezoid Tool," you will make a tool that makes trapezoids and measures their area.

## Exploration 2

Point $U$ is fixed because it is the unit point (trying moving it), but point $P$ and point $Q$ are not fixed. You constructed point $P$ and point $Q$ instead of plotting them exactly, so you can move them along the $x$-axis and see whether you can come up with a better approximation—without constructing more trapezoids.

1. Move point $P$ back and forth in the interval [0, 1], and look at the percentage error that you calculated back in Q2 of Exploration 1—or you can look at the shaded regions above your curve because they also represent your error.

**Q1** For approximately what $x$-value is the error the smallest? (If you are looking at the shading, remember you want the shading to be smallest above all parts of the curve between $x = 0$ and $x = 2$. )

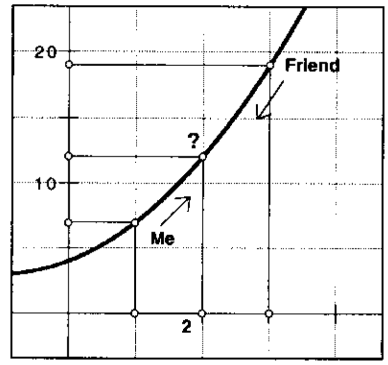



2. Move point $Q$ back and forth in the interval [1, 2], and look at your percentage error from Q2 of Exploration 1.

**Q2** For approximately what $x$-value is the error the smallest?

## What Do You Expect?

**Name(s):** ____________________




Limits are one of the fundamental building blocks of calculus. So what is a limit exactly and how do you find it? This activity will help you answer these questions.

### Sketch and Investigate

Imagine that you're hiking on a trail and it conveniently happens to be curved exactly like the function $q(x) = \frac{x^3 - 8}{x - 2}$, but you don't know that. All you have is an interactive map, similar to the picture at right.

1. **Open** the sketch **Limits.gsp** in the **Exploring Change** folder for your interactive map.

You can also choose **Abscissa(x)** from the Measure menu and then choose **Ordinate(y).**

2. Measure your coordinates and your friend's coordinates by selecting point *Me* and point *Friend*, and then choosing **Coordinates** from the Measure menu.

Now, you're hiking along and you *expect* the trail to be smooth—no potholes or drop-offs. Unfortunately, there's a huge pothole ahead at $x = 2$, but it's not marked on your map, so you don't know about it. What you see and *expect* is to be able to meet your friend, who's hiking down the trail from the other direction to meet you, at the point $x = 2$.

3. Drag point *Me* until you are as close as possible to $x = 2$, but keep your $x$-coordinate less than or equal to 2. Do the same with point *Friend*, but make sure its $x$-coordinate stays greater than or equal to 2.

**Q1** How close to $x = 2$ can you get with each point? What $y$-coordinate did you get for point *Me*? What about for point *Friend*?

**Q2** Based on your points and their movement, what do you *expect* the $y$-coordinate to be at the point $x = 2$?

What you *expect* to see is called a *limit*—actually what you expect to see is called a *left-hand limit* and what your friend expects to see is called a *right-hand limit*. Whether reality agrees with your expectations or not is a different concept altogether, and does not affect your limit.

**Q3** What is the left-hand limit at $x = 2$? What is the right-hand limit?

**Q4** Do both you and your friend expect to see the same $y$-value?

We say that a limit exists if the left-hand limit agrees with the right-hand limit. In this case, you should have gotten $y = 12$ for both parts of Q3. (If you didn't, go back and try again.) What really happens at $x = 2$ is the $y$-value of the function at $x = 2$, or $q(2)$.

## What Do You Expect? (continued)

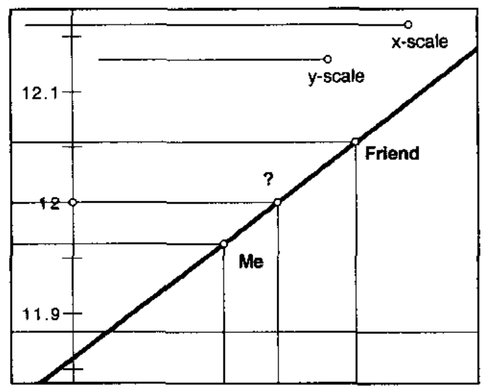
4. Calculate the $y$-value of the function at $x = 2$ by choosing **Calculate** from the Measure menu. Then select the expression for $q(x)$ to enter it into your calculator, enter 2, and close the parentheses.

You should have gotten undefined for $q(2)$. So here reality (the function) does not exist, but that still doesn't affect what you *expect* to see. You still *expect* to see $y = 12$. The key to success with limits is to remember that limits are expectations only—not what really happens!

You may think, "Well, maybe my expectations (or limit) would be different if I zoomed in." Press the *Show Zoom Tools* button. The sliders, *x-scale* and *y-scale,* zoom in the coordinate grid around the point $(a, b)$. In this case $a = 2$ and $b = 12$. Before you start zooming in, make sure that the $x$-coordinates of points *Me* and *Friend* are still as close to $x = 2$ as possible.

If you lose your axes, press the *Show Side Axis* button. If you lose point *Me* or point *Friend,* zoom out until you can see them again.

5. To use both sliders at once, use your **Arrow** tool to select the point at the end of each slider, then click on either point and drag. Familiarize yourself with what the sliders do and then zoom in until point *Me* or point *Friend* is on the edge of your sketch.




If $q(x)$ disappears, zoom out until it reappears. Select the plot and choose **Edit | Properties.** On the Plot panel, change the domain to a small interval around $x = 2$.

6. Drag point *Me* until you are as close as possible to $x = 2$, but keep your $x$-coordinate less than or equal to 2. Do the same with point *Friend,* but make sure its $x$-coordinate stays greater than or equal to 2.

**Q5** How close to $x = 2$ can you get this time? What $y$-coordinate are you at now? What about your friend?

**Q6** What $y$-value do you expect to see now at $x = 2$? (Does your left-hand expectation equal your right-hand expectation?)

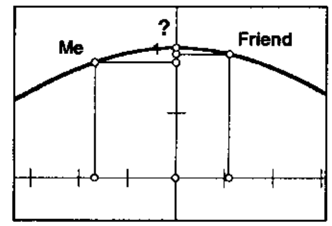
**Q7** Did zooming in strengthen or weaken your expectations?

You could keep zooming in and get closer and closer to $x = 2$ but your expectation, your limit, will stay the same. You then say that the limit of $q(x)$ or the limit of $\dfrac{x^3-8}{x-2}$ as $x$ approaches 2 is 12, or, in shorthand, $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2} = 12$. Notice that the reality, $q(2)$, is still undefined and you can't just substitute $x = 2$ into the function to get this limit.

Let's try another one. In the next few steps, you'll look for the limit of the function $\dfrac{\sin(x)}{x}$ as $x$ approaches 0, or $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x}$. This function definitely has a problem at $x = 0$ and isn't defined there. But it might have a limit just like the above example, and then again, it might not.

7. Go to page 2 in your document. Here $q(x) = \dfrac{\sin(x)}{x}$.

**Q8** Looking at the plot of $q(x)$ at $x = 0$, what $y$-value would you expect?




8. Drag point *Me* until you are as close as possible to $x = 0$, but keep to the left of 0. Do the same with point *Friend*, but make sure it stays to the right of 0.

If $q(x)$ disappears, zoom out until it reappears. Select the plot and choose **Edit | Properties.** On the Plot panel, change the domain to a small interval around $x = 0$.

9. Zoom in and see whether or not your initial guess in Q8 holds up.

**Q9** Does your initial guess hold up? Does the left side agree with the right? What is the limit?

Now let's do a tricky one. In the next few steps, you'll look for the limit of the function $f(x) = 3x + \dfrac{0.01 \cdot |x-2|}{x-2}$ as $x$ approaches 2, or

$\lim\limits_{x \to 2} \left( 3x + \dfrac{0.01 \cdot |x-2|}{x-2} \right)$. This function also definitely has a problem at $x = 2$ and isn't defined there. But it might have a limit just like the other examples—then again, it might not.

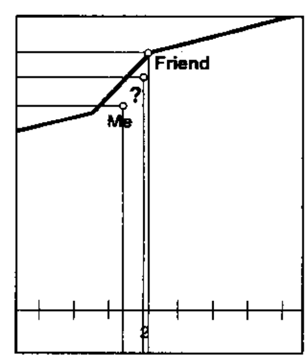
10. Go to page 3 in the document. Here is the plot of your new function.

11. Repeat the process that you used above. Make an initial guess, then drag point *Me* until you are as close as possible to $x = 2$, keeping to the left of 2. Do the same with point *Friend*, but make sure it stays to the right of 2.

**Q10** Does your initial guess hold up? Does the left side agree with the right? What is your expectation at $x = 2$?

If you can't choose the **Properties** box, that means that something other than the function is selected. Click once in an empty area of your sketch, then select the function again.

12. Zoom in at $x = 2$ until your $x$-side axes go from about 1.97 to 2.04. Point *Me* and point *Friend* will look like they "fall off" the function if you take them close to $x = 2$. To take care of that, select the function plot and choose **Properties** from the Edit menu. Choose the Plot panel and change the domain for $f$ to [1.92, 2.08].




**Q11** Try getting as close to $x = 2$ as possible again, using either point. Again make sure point *Me* stays on the left of $x = 2$ and point *Friend* stays on the right.

**Q12** What is the biggest $y$-value you can get on the left-hand side before point *Me* jumps up? (This is the left-hand limit.) Can you ever get a $y$-value of 5.995?

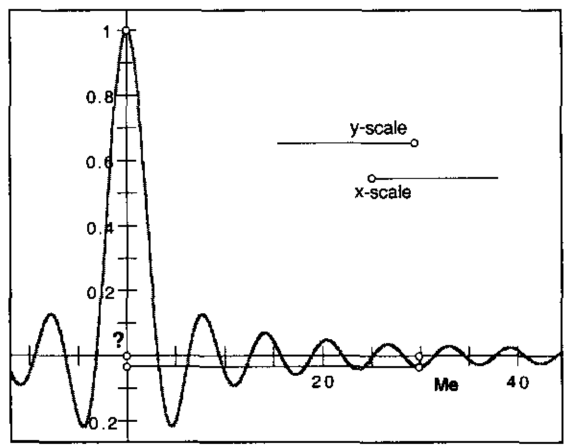
**Q13** What is the smallest $y$-value you can get on the right-hand side before point *Friend* jumps down? (This is the right-hand limit.) Can you ever get a $y$-value of 6.005?

**Q14** Does the left-hand limit agree with the right-hand limit?

Did you get no for Q14? If you didn't, try this again. The function should jump from a $y$-value very close to 5.99 (but a little less than 5.99) to a $y$-value very close to 6.01 (but a little more than 6.01). This function does not have a limit at $x = 2$ because what you expect to see from the left (5.99) is *not* what you expect to see from the right (6.01). The left- and right-hand limits exist, but the limit itself does not. This is like having a canyon across your trail instead of a simple pothole. Potholes can be fixed so that the trail is smooth again. Canyons are a different problem altogether!

### Explore More

Use page 4 to investigate *limits at infinity*. What $y$-value does the function approach if $x$ becomes infinitely huge? Here $q(x)$ is back to the function $\frac{\sin(x)}{x}$. You are going to zoom out this time, so the sketch has been centered at (0, 0).




1. Zoom out until your window goes from –40 to 40 on the $x$-axis.
2. Select point *Me* and slide it along to about $x = 30$.

**Q1** What is your approximate $y$-value at $x = 30$?

**Q2** What is happening to the $y$-values as $x$ gets bigger and bigger?

3. Zoom out until your window goes from –400 to 400 on the $x$-axis and move point *Me* to about $x = 300$.

You might need to zoom in on the $y$-scale to see what is happening around $x = 300$.

**Q3** What is your approximate $y$-value at $x = 300$?

**Q4** What is happening to the $y$-values as $x$ gets bigger and bigger?

If you'd like to zoom out farther you'll need to change the domain of the plot. Select the plot, and then choose **Edit | Properties.** On the Plot panel, change the domain to some values farther out, such as –1000 to 1000.

4. Calculate the value of the function at $x = 1000$ by choosing **Calculate** from the Measure menu and entering $q(1000)$.

**Q5** Based on your calculation and your plot, what do you think the limit is as $x$ becomes infinitely big?