1-5. In the answer space at the right, write the digit 2 four times, positioning the 2s so that the resulting number is as large as possible. (You may not write any symbol other than a 2.) [NOTE: As an example, you could write 222^2; but, by repositioning the 2s, you can write an even larger number.]	1-5.

1-6. What are all ordered triples of positive integers (x,y,z) whose product is 4 times their sum, if $x < y < z$?	1-6.

10/23/07

Problem 1-5

Write a 2. The next 2 can go either to the right (or to the left) of the first, or it can become an exponent. Since $222^2 > 2222$, we must use exponentiation. If the base is 22, $22^{22} > 22^{2^2} = 22^4 > 222^2$. If the base is 2, then the exponent is 222 or 22^2 or 2^{22} or $2^{2^2} = 2^4$, the largest of which is 2^{22}. Finally, $22^{22} < 32^{22} = (2^5)^{22} = 2^{110} < 2^{2^{22}}$, so the largest of all the possibilities is $\boxed{2^{2^{22}}}$.

[**NOTE:** a^{b^c} is understood to mean $a^{(b^c)}$. See, for example, p.28 of *The Lore of Large Numbers* by P. J. Davis (New Mathematical Library) or p.25 of *Mathematics and the Imagination*. In general, $a^{(b^c)} \neq (a^b)^c$. Some calculators don't handle this correctly!]

Problem 1-6

$$\left.\begin{array}{l} xyz = 4(x+y+z) > 4z, \text{ so } xy > 4. \\ xyz = 4(x+y+z) < 12z, \text{ so } xy < 12. \end{array}\right\} \text{Thus, } 5 \le xy \le 11.$$

If $x = 1$, then $5 \le y \le 11$. If $x = 2$, then $3 \le y \le 5$. If $x = 3$, then $y \ge 4$; but this is impossible, since $xy < 12$.

Method I: When $x = 1$, $xyz = 4(x+y+z) \Leftrightarrow yz = 1+y+z$, from which $z = \frac{4(y+1)}{y-4}$. We know $y < z$. The values of $y \in \{5,6,7,8,9,10,11\}$ that work are 5, 6, 8. When $x = 2$, $2yz = 4(2+y+z)$, so $z = \frac{2(y+2)}{y-2}$, so $y = 3$ or 4. Substitute to get z. The 5 solution triples are $\boxed{(1,5,24),\ (1,6,14),\ (1,8,9),\ (2,3,10),\ (2,4,6)}$.

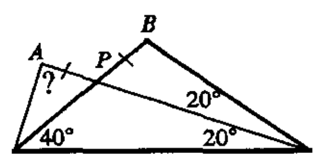
Method II: The two possible x-values, together with the possible xy-values, determine possible (x,y) pairs. Substitute each pair into $xyz = 4(x+y+z)$. See if $z > y$ is also an integer. If $(x,y) = (1,5)$, then $5z = 4(6+z)$, so $z = 24$. Similarly, if $(x,y) = (1,6)$, then $z = 14$; and if $(x,y) = (1,8)$, $z = 9$. No other value of y makes z an integer when $x = 1$. Finally, if $x = 2$ and $(x,y) = (2,3)$, then $6z = 4(5+z)$, so $z = 10$; and if $(x,y) = (2,4)$, then $8z = 4(6+z)$, so $z = 6$. No other value of y makes z an integer when $x = 2$. The 5 solution triples are (1,5,24), (1,6,14), (1,8,9), (2,3,10), (2,4,6).

11/20/07

2-5. I wrote a sequence of n integers. In this sequence, the sum of any 3 consecutive terms is positive, while the sum of any 4 consecutive terms is negative. What is the largest possible value of n?

2-5.

2-6. The degree-measure of each base angle of an isosceles triangle is 40. The bisector of one of the base angles is extended through point P on the leg opposite that base angle so that $PA = PB$, as shown. What is $m\angle A$?

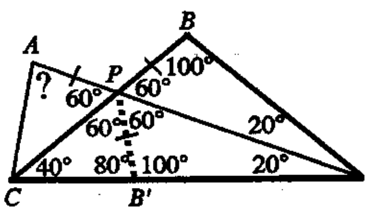


2-6.

Problem 2-5

Some 5-term sequences are −2, −3, 6, −2, −3; or −2, −2, 5, −2, −2; or −3, −3, 7, −3, −3; or −3, −3, 8, −3, −3; or −7, −4, 12, −6, −5. Suppose that c_1, c_2, c_3, c_4, c_5, c_6 is a 6-term sequence in which the sum of any 3 consecutive terms is positive. Then $c_1+c_2+c_3 > 0$, $c_2+c_3+c_4 > 0$, $c_3+c_4+c_5 > 0$, and $c_4+c_5+c_6 > 0$. Adding these 4 inequalities, and rearranging the terms into groups of 4 consecutive terms, we get $(c_1+c_2+c_3+c_4)+(c_2+c_3+c_4+c_5)+(c_3+c_4+c_5+c_6) > 0$. The inequality in the previous sentence cannot be true if the sum of any 4 consecutive terms is negative, so we conclude that there's no such 6-term sequence. Thus, the largest possible value of n is $\boxed{5}$.

Problem 2-6



In the diagram at the right, reflect B to B', reflecting across the line through A and P as shown. This creates 2 new triangles. $\triangle CPA \cong \triangle CPB'$ by SAS, so $m\angle A = m\angle PB'C = \boxed{80 \text{ or } 80°}$.

12/18/07

3-5. The positive integers are placed in groups as follows: (1), (2,3), (4,5,6), (7,8,9,10), (11,12,13,14,15), . . . and so forth, with n consecutive integers in the nth group. What is the first integer in the 100th group?	3-5.

3-6. There's exactly one real number a for which $ax^2+(a+3)x+(a-3)=0$ has two positive integer solutions for x. What are these values of x?	3-6.

Problem 3-5

The first group contains 1 integer, the second contains 2 integers, the third contains 3 integers, . . . , the 99th contains 99 integers. Altogether, the number of integers used in writing all the numbers in the first 99 groups is $1 + \ldots + 99 = (99/2)(1+99) = 4950$. The first integer in the 100th group is $\boxed{4951}$.

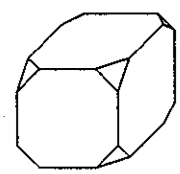
Problem 3-6

Method I: If $a = 0$, only one value of x works. Divide through by $a \neq 0$: $x^2+(\frac{a+3}{a})x+(\frac{a-3}{a}) = 0$, or $x^2 + (1+\frac{3}{a})x + (1-\frac{3}{a}) = 0$†. Call the roots r and s (where r and s are positive integers). Then, $r+s = -1-\frac{3}{a}$ and $rs = 1-\frac{3}{a}$. If we subtract these equations, we will eliminate a, and we'll get $r+s-rs = -2$. (Notice that $s = 1$ is impossible.) Let's solve for r: $s+2 = rs-r = r(s-1)$. Since $s \neq 1$, we can divide through by $(s-1)$, so $r = \frac{s+2}{s-1} = 1+\frac{3}{s-1}$. Since r is a positive integer only when $s-1$ is a positive integral divisor of 3, $s-1 = 1$ or 3, $s = 2$ or 4, and these are the same as the values of r. The only possible positive integral roots x are $\boxed{2, 4}$.

†**Method II:** In $x^2+(1+\frac{3}{a})x+(1-\frac{3}{a}) = 0$ above, let $t = \frac{3}{a}$. We get $x^2+(1+t)x+(1-t) = 0$. This has integral roots. Since it's discriminant is a perfect square, $(1+t)^2-4(1-t) = t^2+6t-3 = (t+3)^2-12$ is a perfect square. This means that t is an integer. Since $(t+3)^2$ is a square and $(t+3)^2-12$ is also a square, let's look for a perfect square, $(t+3)^2$, which is 12 more than another square. That must be 16, which is 12 more than 4. If $(t+3)^2 = 16$, $t = 1$ or -7. When $t = -7$ in the equation on the 2nd line, we get $x^2-6x+8 = 0$. This has the required positive integral solutions.

1/15/08

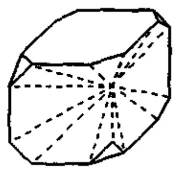
4-5. For what ordered pair of positive integers (b,n), with n as small as possible, is $\log_b n$ the sum of the roots of $9^x - 1006(3^x) + 2008 = 0$?	4-5.

4-6. A cube has all 8 of its corners cut off, leaving the solid shown. If all 24 vertices of the triangles thus formed are connected to each other by line segments, how many of these segments will pass through the interior of this solid?		4-6.

Problem 4-5

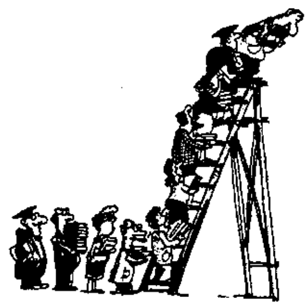
$9^x - 1006(3^x) + 2008 = (3^x - 1004)(3^x - 2) = 0$, so $3^x = 1004$ or $3^x = 2$. Thus, $x = \log_3 1004$ or $x = \log_3 2$. Their sum is $\log_3 2008$, so $(b,n) = \boxed{(3,2008)}$.

Problem 4-6



Each vertex is connected to all 23 other vertices. If we count how many of these 23 line segments lie on the figure's surface, we can subtract to determine how many lie in the figure's interior. Notice that every vertex lies on exactly 2 of the cube's original faces. Pick any face. Call one of its 8 vertices A. On that face, line segments connect A to each of the other 7 vertices. But vertex A appears on 2 faces, and segments connect A to the 2nd face's other 7 vertices. The total number of segments on the 2 faces is not $7+7 = 14$, because we counted the common edge of the 2 faces as 2 segments, 1 in each face. So, there are only 6 new segments on the 2nd face. That's a total of $7+6 = 13$ line segments on the surface. Since each vertex is connected to all 23 other vertices, $23-13 = 10$ of the line segments drawn from each vertex will pass through the figure's interior. Each of the $24 \times 10 = 240$ segments has 2 endpoints, so each has been counted twice, once for each endpoint. The actual number of segments is $\boxed{120}$.

2/12/08

5-5. If 26 people (whose first names each start with a different letter) line up at random to climb a ladder, what is the probability that Pat is lucky enough to stand next to Dale? 	5-5.

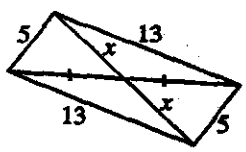
5-6. Two sides of a triangle have lengths 5 and 13. The median to the third side has length x. What are all possible integral values of x?	5-6.

Problem 5-5

Method I: Line up everyone *except* Pat. There are 24 slots between the 25 people already on line. In addition, Pat can also go to the front or the rear of the line. Of these 26 possible positions for Pat, 2 are next to Dale, so the required probability is $\boxed{\frac{1}{13}}$.

Method II: Treating Pat-Dale as a single entity (with 2 arrangements), the probability is $\frac{2!\,25!}{26!} = \frac{1}{13}$.

Problem 5-6



Form a parallelogram whose side-lengths are 5 and 13. Any of its half-diagonals is the required median. Let the median's length be x, as shown. Two triangles shown have side-lengths 5, 13, and $2x$. Apply the triangle inequality to either one to get $13-5<2x<13+5$. Divide by 2 to get $4<x<9$. The answers are $\boxed{5, 6, 7, 8}$.

3/18/08

6-5. If a, b, and c are positive integers, what is the largest value less than 1 representable by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$?	6-5.

6-6. Bricklayers Pat and Lee alternate turns in a game in which each player removes from 1 to 100 bricks from a common pile that initially has 2008 bricks. The player taking the last brick *wins*. If Pat and Lee both play perfectly, and if Pat goes first, then how many bricks must Pat take on his first turn to guarantee a win? 	6-6.

Problem 6-5

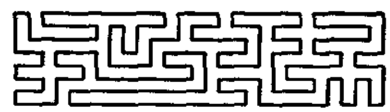
To meet the criteria, clearly $a,b,c \geq 2$. Without loss of generality, we can assume $a \leq b \leq c$. To maximize the sum of their reciprocals (keeping this sum < 1), we want to *minimize* a,b,c. If $a = 2$, then $\frac{1}{a} = \frac{1}{2}$. Since the sum of all three reciprocals < 1, the sum of the reciprocals of b and c must be less than $\frac{1}{2}$. Thus, both $b,c \geq 3$. If $b = 3$, then $\frac{1}{2}+\frac{1}{3}+\frac{1}{c} < 1$. Therefore, $\frac{1}{c} < \frac{1}{6}$, so set $c = 7$, and $\frac{1}{2}+\frac{1}{3}+\frac{1}{7} = \boxed{\frac{41}{42}}$. Continuing, if $a = 2$ and $b = 4$, then $c > 4$, so use $c = 5$. The sum is $\frac{1}{2}+\frac{1}{4}+\frac{1}{5} = \frac{19}{20} < \frac{41}{42}$. If $b = 5$, then the sum $< \frac{9}{10} < \frac{41}{42}$. If $a = 3$, then the maximum sum is $\frac{1}{3}+\frac{1}{3}+\frac{1}{4} = \frac{11}{12} < \frac{41}{42}$.

Problem 6-6

On the next-to the last round, if Pat leaves 101 bricks, then no matter how many bricks Lee takes (which must be a whole number from 1 to 100 inclusive), Pat can take all the remaining bricks, since the number of bricks remaining will be 100 or less. Similarly, on the round before that, if Pat leaves 202 bricks, then no matter how many bricks Lee takes, Pat can take enough bricks so that 101 bricks remain in the pile at Lee's turn. In general, Pat's winning strategy should be to take enough bricks each round so that the number of bricks remaining is a multiple of 101. If Pat leaves a multiple of 101 bricks each time, then eventually Pat will leave exactly 101 bricks. On Pat's first turn, Pat must remove enough bricks so that the number left is a multiple of 101. Since 2008 is 89 more than a multiple of 101, the number of bricks that Pat should take on the first round is $\boxed{89}$.

[**NOTE:** If the player taking the last brick loses, then, working backwards, Pat must leave 1, 102, 203, . . . , $101n+1$, . . . , 1819, 1920 bricks. Now, Pat's winning strategy is to leave Lee with $101n+1$ bricks. If Lee takes x bricks, $1 \leq x \leq 100$, Pat should take $101-x$ bricks. Eventually, this leaves Lee with 1 brick. In this case, Pat must first take 88 bricks.]

1-5. Write $4x^2-9y^2+4x^3+6x^2y$ as a product of two non-constant polynomials with integral coefficients.	1-5.

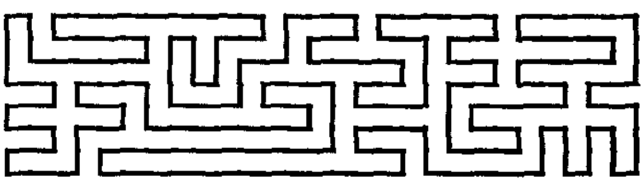
1-6. When polygon P, seen at the right, is drawn on a 7×27 grid of unit squares, it passes once through every vertex of every unit square in the grid. What is P's perimeter? 	1-6.

Problem 1-5

$4x^2 - 9y^2 + 4x^3 + 6x^2y = (4x^2 - 9y^2) + (4x^3 + 6x^2y) =$
$(2x-3y)(2x+3y) + (2x^2)(2x+3y) = \boxed{(2x+3y)(2x-3y+2x^2)}$.

Problem 1-6

Let's consider a similar problem applied to a much smaller polygon, say, a 1×1 grid of unit squares, where the polygon is a single unit square. The number of vertices of the unit square is $(1+1)(1+1)$. If there were a 1×2 rectangle, made of 2 unit squares, then the number of vertices would be $(1+1)(2+1) = 2 \times 3 = 6$. Similarly, a 7×27 grid of unit squares will have $(7+1)(27+1) = 8 \times 28 = 224$ vertices. Begin at any of the polygon's vertices and traverse the perimeter of P. You will travel 1 unit each time you go to a new vertex, and the final unit is the return to the first vertex, completing the polygon (which is a closed figure). Hence, the perimeter is $\boxed{224}$.



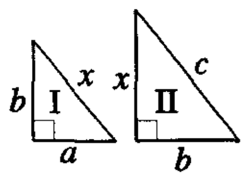
2-5. If exactly two different linear functions, f and g, satisfy $f(f(x)) = g(g(x)) = 4x+3$, what is the product of $f(1)$ and $g(1)$?	2-5.

2-6. Two triangles which are *not* congruent can actually have five pairs of congruent parts! For example, triangles with side-lengths 8, 12, 18 and 12, 18, 27 have five pairs of congruent parts (two sides and three angles). If two non-congruent *right triangles* have five pairs of congruent parts, what is the ratio of the length of the hypotenuse of either triangle to the length of that triangle's shorter leg?	2-6.

Problem 2-5

If $f(x) = ax+b$ is the linear function f, then $f(f(x)) = a(ax+b)+b = 4x+3$, so $a^2x+ab+b = 4x+3$. Equate coefficients of x to get $a^2 = 4$, from which $a = \pm 2$. Equate constant terms to get $ab+b = 3$. Each value of a gives rise to a value of b. Then, $(a,b) = (2,1)$ and $(-2,-3)$, so let $f(x) = 2x+1$ and $g(x) = -2x-3$. Thus, $f(1) = 3$, $g(1) = -5$, and their product is $\boxed{-15}$.

Problem 2-6

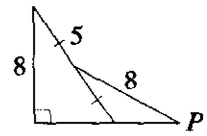


The angles are $\cong$*, so the $\triangle$s are similar. Thus, $\frac{a}{b} = \frac{b}{x}$, or $b^2 = ax$. In $\triangle$I, $a^2+b^2 = x^2$, from which $a^2+ax = x^2$. Since $x > 0$ and $a > 0$, we first use the quadratic formula to solve for x, then we can divide by a. Solving, we get $x = \frac{a + \sqrt{a^2 + 4a^2}}{2}$, so $\frac{x}{a} = \boxed{\frac{1+\sqrt{5}}{2}}$.

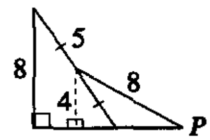
*[**NOTE**: Of the 5 pairs of congruent parts, either 3 are angles or 3 are sides. If the 3 congruent pairs were sides, then the triangles would be congruent, and the 3 pairs of angles would also be congruent then. It must be the case that 3 pairs of angles are congruent, but only 2 pairs of sides are congruent.]

[**NOTE**: The answer is known as the *Golden Ratio*.]

[**NOTE**: The hypotenuse of the larger triangle is longer than any side of the smaller triangle, so the legs of the larger triangle are as long as the hypotenuse and longer leg of the smaller triangle. The larger triangle's shorter leg is as long as the smaller triangle's longer leg.]

3-5. Line segment ℓ connects the midpoint of the hypotenuse of a 6-8-10 right triangle to a point P on the extension of the triangle's shorter leg, as shown. If the length of ℓ is 8, what is $m\angle P$? 	3-5.
3-6. The coefficients of polynomial P are non-negative integers. If $P(1) = 6$ and $P(5) = 426$, what is the value of $P(3)$?	3-6.

Problem 3-5



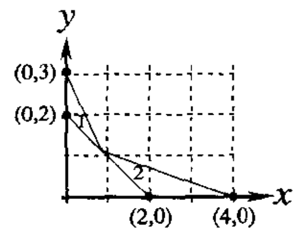
Drop a perpendicular from the midpoint of the hypotenuse of the right triangle to the shorter leg. Since this perpendicular passes through the midpoint of the hypotenuse and is parallel to the longer leg, that perpendicular is half as long as the longer leg. We created a right triangle whose hypotenuse is 8 and whose shorter leg (the perpendicular we dropped) is 4. Thus, $m\angle P = \boxed{30}$ or 30°.

Problem 3-6

If P were a polynomial with non-negative integral coefficients of degree ≥ 4, then $P(5) \geq 5^4 = 625$. Since $P(5) = 426 \neq 625$, we know that P can't be such a polynomial. Let $P(x) = ax^3+bx^2+cx+d$, where some (not all) of the coefficients are 0. Since $P(1) = a+b+c+d = 6$, if any coefficient were 6, the others would be 0. But, when $x = 5$, $P(5) = 6x^n = 6(5^n) = 426$ is impossible when n is an integer. Thus, every coefficient ≤ 5. Can any coefficient be 5? If one coefficient were 5 and another were 1, then the others would be 0. If $a = 5$, then $P(5) > 5x^3 = 625$. This is impossible since $P(5) = 426$. If another coefficient is 5, then $P(x) \leq x^3+5x^2$, so $P(5) \leq 5^3+5(5^2) = 250$. Therefore, no coefficient exceeds 4. Since $426 = 3\times125+2\times25+1$, $(a,b,c,d) = (3,2,0,1)$. This is the only 4-tuple of non-negative integers that satisfies the requirements $P(1) = 6$ and $P(5) = 426$, since base 5 representation is unique if every coefficient ≤ 4; so $P(x) = 3x^3+2x^2+1$, and $P(3) = \boxed{100}$.

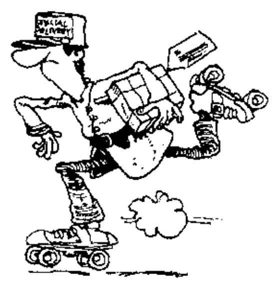
1/13/09

4-5. In the diagram shown at the right, there's exactly one point on the segment connecting (0,2) to (2,0) for which the angles marked 1 and 2 will be congruent. What are the coordinates of this point?



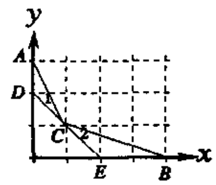
4-5.

4-6. Starting from opposite ends of a street, each of two messenger boys skated at his top speed towards the other's starting point. From the time they passed each other, one messenger took 1 more minute, and the other took 2 more minutes, to reach their respective destinations. How many minutes did it take the faster messenger to skate the entire distance?



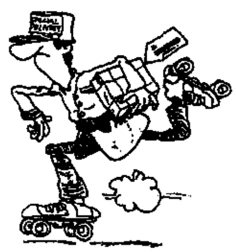
4-6.

Problem 4-5



In the diagram seen at the right, $\triangle ACD \sim \triangle BCE$, so $AD{:}BE = 1{:}2 = DC{:}CE$. Thus DC is $\frac{1}{3}$ of DE, so we can conclude that the x-coordinate of C is $\frac{1}{3}$ of the x-coordinate of E. Similarly, the y-coordinate of C is $\frac{2}{3}$ of the y-coordinate of D. The coordinates of point C are $\boxed{\left(\frac{2}{3}, \frac{4}{3}\right)}$.

Problem 4-6



Method I: If t represents the number of minutes each boy skates before passing the other boy, then the slower one took $t+2$ minutes and the faster $t+1$ minutes to travel 1 street-length. Their respective rates, in street-lengths per minute, are $\frac{1}{t+2}$ and $\frac{1}{t+1}$. The total distance traveled by the two boys during the first t minutes is 1 street-length, so $\frac{t}{t+2} + \frac{t}{t+1} = 1$. The positive solution of this equation is $t = \sqrt{2}$, so $t+1 = \boxed{\sqrt{2}+1}$.

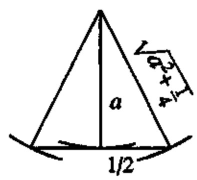
Method II: Suppose the faster messenger skates at the rate of r_1 street-lengths per minute and the slower messenger skates at the rate of r_2 street-lengths per minute. The distance each skated before they met equals the distance the other skated after they met. So $tr_1 = 2r_2$ and $tr_2 = 1r_1$, and we have that $2r_2 = tr_1 = t^2 r_2$. Thus, $t = \sqrt{2}$ and the faster messenger took $\sqrt{2}+1$ minutes to skate the entire distance.

5-5. One circle is inscribed in, and a second circle is circumscribed about, a regular polygon of 2009 sides. If the polygon's perimeter is 2009, what is the area of the region between the two circles?	5-5.

5-6. All students at the Academy of Music and Math take both music and math. The probability that a student has an A in math is 1/6. The probability that a student has an A in music is 5/12. The probability that a student with an A in math has an A in music plus the probability that a student with an A in music has an A in math is 7/10. What is the probability that a student has A's in both subjects?	5-6.

Problem 5-5

Method I: Let the length of the polygon's apothem (a radius of the inscribed circle) be a. Each side of the polygon has length 1. Use the Pythagorean Theorem to find the length of



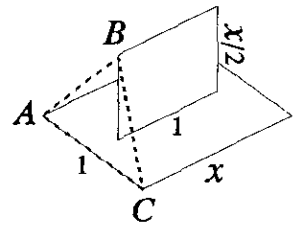
a radius of the larger circle (the circumscribed circle), as shown. The area of the larger circle is $\pi(a^2+\frac{1}{4})$ and the area of the smaller circle is πa^2, so the area between the two circles is their difference, $\boxed{\frac{\pi}{4}}$.

Method II: Relabel the lengths in the above diagram so the radii are R and r, with $R > r$. In the right $\triangle$, $R^2-r^2 = \frac{1}{4}$. To get the difference in the areas of the two circles, multiply this equation through by π.

Problem 5-6

Let the number of students at the Academy be n. Since the probability that a student has an A in math is $\frac{1}{6}$, only $\frac{n}{6}$ students have an A in math. Since the probability that a student has an A in music is $\frac{5}{12}$, only $\frac{5n}{12}$ students have an A in music. The probability that a student with an A in math has an A in music plus the probability that a student with an A in music has an A in math is $\frac{7}{10}$. If x is the number of students who have an A in both, then $\frac{x}{n/6} + \frac{x}{5n/12} = \frac{7}{10}$, so $x = \frac{n}{12}$. Since the number of students who have A's in both is $\frac{n}{12}$, and since the total number of students is n, the probability that a student has A's in both subjects is $\boxed{\frac{1}{12}}$.

6-5. Two rectangles are positioned in space so that the smaller one is perpendicular to the larger, as shown. A longer side of the smaller rectangle is parallel to and midway between both longer sides of the larger, and is also equidistant from both shorter sides, as shown. The lengths of the sides of the larger rectangle are 1 and x; while those of the smaller are 1 and $x/2$, where $x > 1$. For what value of x will $\triangle ABC$ be equilateral?

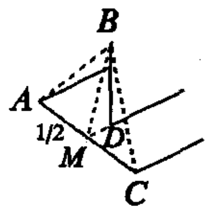


6-5.

6-6. If $3\cos x + 4\cos y = 5$ what is the greatest possible value of
$$3\sin x + 4\sin y?$$

6-6.

Problem 6-5



In right $\triangle BDM$, $BD = \frac{x}{2}$, and MD is half the difference of the lengths of the two rectangles, so $MD = \frac{x-1}{2}$ and $MB^2 = \left(\frac{x-1}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = \frac{2x^2-2x+1}{4}$. Each side of $\triangle ABC$ will have a length of 1 only if $MB^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} = \frac{2x^2-2x+1}{4} \Leftrightarrow 2x^2-2x-2 = 0$. Finally, since x is positive, $x = \boxed{\frac{1+\sqrt{5}}{2}}$.

[**NOTE**: The answer is known as the *Golden Ratio.*]

Problem 6-6

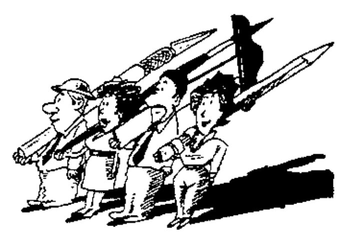
Let $3\sin x + 4\sin y = A$. Since $3\cos x + 4\cos y = 5$, square both equations and add results: $A^2+25 = 9\sin^2 x+24\sin x\sin y+16\sin^2 y+9\cos^2 x+24\cos x\cos y+16\cos^2 y = 25+24(\cos x\cos y+\sin x\sin y)$, so $A^2 = 24\cos(x-y)$. At most, $A^2 = 24$ and $A = \boxed{\sqrt{24}}$.

[**NOTE**: If $x = y$, $3\cos x + 4\cos y = 3\cos x + 4\cos x = 7\cos x = 5$, so $x = y = \pm\text{Arc}\cos\frac{5}{7}$. Choosing $x > 0$, $\cos x = \frac{5}{7}$, $\sin x = \frac{\sqrt{24}}{7}$, & $3\sin x + 4\sin y = 7\sin x = \sqrt{24}$. This function's actual and theoretical maximum is $\sqrt{24}$.]

6-5. A function f will be called *repetitive* if there are at least two different values of x in the interval $0 \leq x \leq 1$ for which $f(x)$ has the same value. What are all real numbers b for which $f(x) = x^2 + bx + 3$ is repetitive?

6-5.

6-6. Our design team drew all 48 line segments that connect 6 points on one side of rectangle R to 8 points on the opposite side. In at most how many different points inside R can these 48 line segments intersect?



6-6.

Problem 6-5

The graphs of $y = x^2+bx+3$ and $y = x^2$ are translations of each other, so it's easy to see that the graph of $y = x^2+bx+3$ will be *repetitive* when $0 \le x \le 1$ if and only if its minimum occurs when $0 < x < 1$. This parabola's axis of symmetry is $x = -\frac{b}{2}$. Finally, $0 < -\frac{b}{2} < 1$ is equivalent to $\boxed{-2 < b < 0}$.

Problem 6-6

Any 4 points, 2 from one side of the rectangle and 2 from the opposite side, determine a convex quadrilateral whose diagonals cross once inside R. The total number of points of intersection that lie inside R equals $\binom{6}{2} \times \binom{8}{2} = 15 \times 28 = \boxed{420}$.

[**NOTE:** If some of these intersection points are coincident, then there can be fewer than 420 distinct points of intersection.]

1-5. In a certain two-person game, each player, in turn, removes 1, 2, 3, 4, or 5 toothpicks from a common pile, until the pile is exhausted. The player who takes the last toothpick loses. If the starting pile contains 300 toothpicks, how many toothpicks must the first player take on the first turn to guarantee a win with perfect subsequent play?

1-5.

1-6. Last year, each of Big Al's five brothers gave a gift of money to Big Al. The dollar amounts were consecutive integers, and their sum was a perfect cube. If the brothers decide to give Big Al cash gifts with those same properties both this year and next year as well, but this year's sum is larger than last year's, and next year's sum larger still, what is the least possible dollar amount Big Al could get next year from his five brothers combined?

1-6.

Problem 1-5

Work backwards. You want to leave your opponent with 1 toothpick and avoid being left with 1 yourself. You can't leave your opponent with 2, 3, 4, 5, or 6 toothpicks or he will be able to remove the right number to leave you with 1 toothpick. It would be great to leave him with 7, because whatever he does from there, you are safe and you will be able to leave him with 1. Now treat leaving your opponent with 7 as the goal; the same logic says that leaving him with 8, 9, 10, 11, or 12 would be bad, but 13 is a good number to leave at the end of your turn. The goal is to leave your opponent with $6n+1$ toothpicks (since that eventually leaves him with $6(0)+1 = 1$ toothpick). Once you've left him with $6n+1$ toothpicks, do this: whenever your opponent takes x toothpicks, you take $6-x$ toothpicks. The largest integer less than 300 that fits the goal is $6(49)+1 = 295$, so you should remove $300-295 = \boxed{5}$ toothpicks at your first turn.

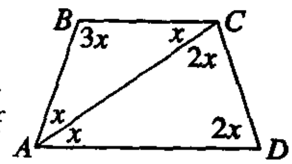
Problem 1-6

Since the sum of these 5 consecutive positive integers is a perfect cube, $(x-2)+(x-1)+x+(x+1)+(x+2) = 5x$ is a perfect cube. This is possible only if x is of the form $25n^3$, where n is a positive integer. The smallest possible value of n is 1. Last year, the least amount Al could have possibly received was $5\times\$25 = \$125$. This year, the least amount of money he could receive as a gift is $5\times\$(25\times 2^3) = \$1000$. Next year, the least dollar amount that Al can possibly receive is $5\times(25\times 3^3) = 3^3\times 5^3 = 15^3 = \boxed{3375}$ or \$3375.

2-5. Trapezoid $ABCD$ is isosceles, with $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \cong \overline{BC} \cong \overline{CD}$. If $\overline{AC} \cong \overline{AD}$, what is $m\angle ADC$?	2-5.

2-6. Al and Bo, who together have \$168, bet against each other. Each bets the same fraction of his money as the other bets. If Al wins, he'll have double what Bo then has. If Bo wins, he'll have triple what Al then has. How many dollars does Al have at the start?	2-6.

Problem 2-5



Let $x = m\angle BAC = m\angle BCA$. Alternate-interior angles of parallel lines are congruent, so $m\angle CAD = x$ too. Now, $m\angle CDA = m\angle BAD = 2x$. Finally $m\angle B = m\angle BCD = 3x$. Then, $5x = 180$, so $x = 36$ and $m\angle ADC = 2x = \boxed{72}$ or 72°.

Problem 2-6

Method I: Let Al have $\$a$ and let Bo have $\$b$. Let k be the fraction of his money that each bets, so the actual bets are $\$ka$ and $\$kb$. Since $a+kb = 2(b-kb)$, we know that $b = \frac{a}{2-3k}$. Also, $b+ka = 3(a-ka)$, so $a = \frac{b}{3-4k}$. Thus, $b = \frac{a}{2-3k} = \frac{b}{3-4k} \div (2-3k)$, so $(2-3k)\times(3-4k) = 1$. Therefore, $k = 1$ (impossible) or $k = \frac{5}{12}$. Finally, $168-a = \frac{a}{2-3k}$, so $a = \boxed{72}$.

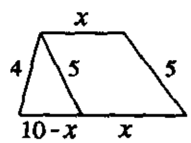
Method II: If Al wins, he'll have twice as much as Bo. Thus, $a+kb = (2/3)(168) = 112$ and $b-kb = 168 - 112 = 56$. If Bo wins, he'll have 3 times as much as Al, so $b+ka = (3/4)(168) = 126$ and $a-ka = 168 - 126 = 42$. Therefore, $238 = 112 + 126 = a+kb + b+ka = a+b + k(a+b) = (a+b)(1+k)$. We were told that $a+b = 168$, so $1+k = \frac{238}{168} = \frac{17}{12}$ and $k = \frac{5}{12}$. Now, $a-ka = a(1-k) = 42$, so $\frac{7}{12}a = 42$, and $a = 72$.

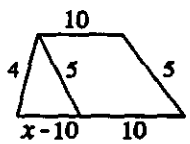
Method III: $3kb = 2b-a$ and $4ka = 3a-b$. Multiplying the first equation by $4a$, $12kab = 8ab-4a^2$. Multiplying the second equation by $3b$, $12kab = 9ab-3b^2$. Subtracting, rearranging, and factoring, $4a^2+ab-3b^2 = (4a-3b)(a+b) = 0$. Thus, $a = \frac{3}{4}b$. Since $a+b = 168$, $a = 72$.

3-5. The bases of trapezoid T have lengths 10 and x, and the legs have lengths 4 and 5. What are the four positive integers less than 19 which cannot be a value of x?	3-5.

3-6. The 6-digit number $n = ABCDE6$ ends in a 6. Transferring the 6 from last place to first place, and leaving the other digits unchanged relative to one another, results in the same number that one would get by multiplying the original number by 4. Thus, $4 \times ABCDE6 = 6ABCDE$. What is the value of the 6-digit number $n = ABCDE6$?	3-6.

Problem 3-5





First, $x \neq 10$. (If $x = 10$, then the quadrilateral would be a parallelogram—but a parallelogram cannot have opposite sides with lengths 4 and 5.) The diagrams shown above represent the two possible situations. The resulting inequalities and their solutions are:

$$\begin{array}{ll} 1 < 10-x < 9 & 1 < x-10 < 9 \\ -9 < x-10 < -1 & 11 < x < 19 \\ 1 < x < 9 & \end{array}$$

The only positive integers < 19 which are not among these solutions are $\boxed{1, 9, 10, 11}$.

Problem 3-6

Method I: We know that $4(ABCDE6) = 6ABCDE$, so $4(10ABCDE+6) = 600\,000+ABCDE$, from which $39ABCDE = 599\,976$. Dividing by 39, $ABCDE = 15384$, so $ABCDE6 = \boxed{153\,846}$.

[**NOTE:** Method II shows that the problem may be solved without knowing how many digits n has.]

Method II: The numbers are of the form $4(\ldots 6) = 6\ldots$. Start by multiplying: $4(\ldots 6) = \ldots 4$, with a carry of 2. Put the 4 to the left of the 6 to get a new trial multiplier, $(\ldots 46)$, which will be multiplied by 4. To multiply $4(\ldots 46)$, we continue by multiplying 4×4, then adding 2 (the carry from 4×6) to get 18. Put the 8 in front to get $(\ldots 846)$, and the carry is 1. Continue this process until there's no carry. In this problem, there's no carry at $4\times 153\,846 = 615\,384$.

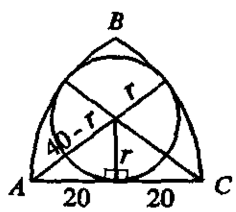
4-5. Two knights simultaneously began to ride towards each other. They first passed each other after 30 seconds, continued to each other's starting point, lost no time in turning, and returned towards each other, each retracing his earlier path. If each knight rode at a steady speed, how many seconds after their first pass did they pass each other for the second time? 	4-5.

4-6. Equilateral Gothic arch ABC is made by drawing line segment AC, circular arc AB with center C, and circular arc BC with center A. A circle inscribed in this Gothic arch is tangent to $\widehat{AB}$, $\widehat{BC}$, and $\overline{AC}$. If $AC = 40$, what is the area of the circle?	4-6.

Problem 4-5

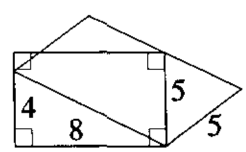
When the knights first pass each other, the total of the distances they've traveled equals the distance that had been between them at the start. By the time they pass each other a second time, the total of all the distances they've traveled then equals three times the distance that had been between them at the start. In other words, the total distance covered to get from the first point at which they pass each other to the second is twice the sum of the distances traveled by the two knights to get from their respective starting points to the point at which they passed each other the first time. It took 30 seconds for them to pass each other the first time, so the number of seconds between the first and second times they passed each other is $\boxed{60}$ or 60 seconds.

Problem 4-6



In the diagram, let a radius of the inscribed circle be r. $\overline{AC}$, the radius of both circular arcs, is 40, so the longer leg of each right triangle is 20, and the hypotenuse is $40-r$. Use the Pythagorean Theorem in either right triangle to get $r^2+20^2 = (40-r)^2$. Solving, we get $r = 15$; so the area of the inscribed circle is $\boxed{225\pi}$.

5-5. Let f and g be functions with respective inverses F and G. An equation relating f and g is $3f(x) = 2g(x)$. If all four functions are defined for all real numbers, and if $F(2010) = G(n)$, what is the value of n?	5-5.

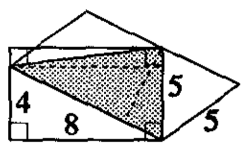
5-6. As shown, a rectangle and a parallelogram have one vertex in common, and a second vertex of each lies on a side of the other. If each quadrilateral has two sides of length 5, what is the area of the parallelogram? 	5-6.

Problem 5-5

Method I: Since f and F are inverses, $F(2010) = t$ if and only if $f(t) = 2010$. Since $3f(t) = 2g(t)$, it follows that $g(t) = (3/2)f(t)$, so $g(t) = 3015$. Now apply G to both sides to get $t = G(3015)$, so $n = \boxed{3015}$.

Method II: Let $k = 3f(x) = 2g(x)$. Then $f(x) = k/3$. Applying the inverse F to each side, $x = F(k/3)$. Similarly, $g(x) = k/2$, so $x = G(k/2)$. Equating results, $F(k/3) = G(k/2)$. Thus, $F(2010) = F(6030/3) = G(6030/2) = G(3015)$, from which $n = 3015$.

Problem 5-6

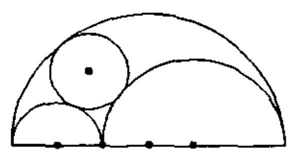


A side of the shaded triangle is a side of the rectangle, and one altitude of the shaded triangle is an altitude of the rectangle, so the area of the shaded triangle is half the area of the rectangle. In the same way, one side of the shaded triangle is a side of the parallelogram, and the shaded triangle and the parallelogram have the same altitude, so the shaded triangle's area is half the parallelogram's area. Therefore, the area of the parallelogram must equal the area of the rectangle $= 5 \times 8 = \boxed{40}$.

[**NOTE 1:** As long as the rectangle's and parallelogram's shorter sides are the same length, then a rotation of the parallelogram can be made around the vertex angle of the isosceles triangle so that its side that's marked with a 5 coincides with the rectangle's side that's marked with a 5. This rotation through an angle equal to the measure of the isosceles triangle's vertex angle will then place the parallelogram's other side of length 5 on the very line that contains the rectangle's left side.]

[**NOTE 2:** Proving that the measure of the isosceles triangle's vertex angle is exactly $2 \arctan \frac{1}{2}$ is a problem in elementary geometry, even though the language used is trigonometry. Just show that one angle is twice another. Now **that's** a nice problem!]

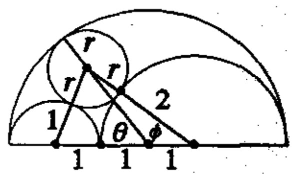
6-5. What is the least possible integer that can be the sum of an infinite geometric progression whose first term is 10?	6-5.

6-6. Semicircles with radii 1 and 2 are externally tangent to each other and internally tangent to a semicircle of radius 3, in the manner shown. How long is a radius of the circle tangent to all three semicircles, as shown? 	6-6.

Problem 6-5

Call the progression's sum S and call its common ratio r, where $|r| < 1$. Since $a_1 = 10$, we have $S = \frac{10}{1-r}$. Since $|r| < 1$, $S > \frac{10}{1-(-1)} = 5$. The least integer > 5 is $\boxed{6}$.

Problem 6-6



Since the radius of the large circle is 3, the cevian through the interior of the large triangle is $3-r$. In the triangle with angle θ, $(1+r)^2 = 2^2+(3-r)^2-2(2)(3-r)\cos\theta$ follows from the law of cosines. In the triangle with angle ϕ, we get $(2+r)^2 = 1^2+(3-r)^2-2(1)(3-r)\cos\phi$. Double both sides of this equation, then use the fact that, since angles θ and ϕ are supplementary, $\cos\phi = -\cos\theta$. We then get $2(2+r)^2 = 2+2(3-r)^2+2(2)(1)(3-r)\cos\theta$. Add this to the equation derived from the first triangle to get $2(2+r)^2+(1+r)^2 = 6+3(3-r)^2$. Solving, $r = \boxed{\frac{6}{7}}$.